MIN-PLUS METHODS IN EIGENVALUE PERTURBATION THEORY AND GENERALISED LIDSKIĬ-VIŠIK-LJUSTERNIK THEOREM

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Abstract. We extend the perturbation theory of Višik, Ljusternik and Lidskiĭ for eigenvalues of matrices, using methods of min-plus algebra. We show that the asymptotics of the eigenvalues of a perturbed matrix is governed by certain discrete optimisation problems, from which we derive new perturbation formulæ, extending the classical ones and solving cases which were singular in previous approaches. Our results include general weak majorisation inequalities, relating leading exponents of eigenvalues of perturbed matrices and min-plus analogues of eigenvalues.

1. Introduction

Let $\mathcal{A}_\epsilon$ denote a $n \times n$ matrix whose entries, which are continuous functions of a parameter $\epsilon > 0$, satisfy

$$(\mathcal{A}_\epsilon)_{ij} = a_{ij}\epsilon^{A_{ij}} + o(\epsilon^{A_{ij}}) \tag{1}$$

when ϵ goes to 0, where $a_{ij} \in \mathbb{C}$, and $A_{ij} \in \mathbb{R} \cup \{+\infty\}$. (When $A_{ij} = +\infty$, this means by convention that $(\mathcal{A}_\epsilon)_{ij}$ is identically zero.) The goal of this paper is to give first order asymptotics

$$\mathcal{L}^i_\epsilon \sim \lambda_i \epsilon^{\Lambda_i} \enspace ,$$

with $\lambda_i \in \mathbb{C} \setminus \{0\}$ and $\Lambda_i \in \mathbb{R}$, for each of the eigenvalues $\mathcal{L}^1_\epsilon, \dots, \mathcal{L}^n_\epsilon$ of $\mathcal{A}_\epsilon$.

Computing the asymptotics of spectral elements is a central problem of perturbation theory, see [Kat95] and [Bau85]. For instance, when the entries of $\mathcal{A}_\epsilon$ have Taylor (or, more generally, Puiseux) series expansions in ϵ, the eigenvalues $\mathcal{L}^i_\epsilon$ have Puiseux series expansions in ϵ, which can be computed by applying the Newton-Puiseux algorithm to the characteristic polynomial of $\mathcal{A}_\epsilon$. The leading exponents Λ_i of the eigenvalues of $\mathcal{A}_\epsilon$ are the slopes of the associated Newton polygon: the difficulty is to determine these slopes from $\mathcal{A}_\epsilon$.

The case of a linear perturbation of degree one

$$\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b \enspace , \qquad \mathcal{A}_0, b \in \mathbb{C}^{n\times n} \enspace ,$$

has been particularly studied. It suffices to consider the case where $\mathcal{A}_0$ is nilpotent, which is the object of a theory initiated by Višik and Ljusternik [VL60] and completed by Lidskiĭ [Lid65]. Their result shows that for generic values of the entries of b, the exponents Λ_i are the inverses of the dimensions of the Jordan blocks of

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$\mathcal{A}_0$. Then, the coefficients λ_i can be obtained from the eigenvalues of certain Schur complements built from the matrices $\mathcal{A}_0$ and b.

However, the construction of Višik, Ljusternik and Lidskiĭ has many singular cases, in which the Schur complements do not exist, and so, their approach does not apply to non-generic situations, such as the case when the matrix b has a sparse or structured pattern.

The problem of generalising the theorem of Višik, Ljusternik and Lidskiĭ, i.e., of "categorising all possible behaviours as a function of the perturbation b", to quote the introduction of the article of Ma and Edelman [ME98], has received much attention. Their article solves cases where $\mathcal{A}_0$ and b have certain Jordan and Hessenberg structures, respectively. This problem is also considered in the survey of Moro, Burke, and Overton [MBO97], which includes a slight refinement of Lidskiĭ's result together with an extension in special cases. Similar problems have been raised for matrix pencils, see in particular Najman [Naj99]. See also Edelman, Elmroth and Kågström [EEK97, EEK99] for a geometric point of view. Numerical motivation can be found there, as well as in the theory of pseudospectra, see Trefethen and Embree [TE05] for an overview. In [Mur90], Murota gave an algorithm to compute the Puiseux series expansions of the eigenvalues of a matrix whose entries are given by polynomials (or even formal series) in some indeterminate. This algorithm sheds some light on the problem raised by Ma and Edelman (although this problem is not considered there).

In this paper, we use min-plus algebra to give elements of answer to the problem raised by Ma and Edelman.

To describe our results, let us recall that the min-plus semiring, $\mathbb{R}_{\min}$, is the set $\mathbb{R} \cup \{+\infty\}$, equipped with the addition $(a, b) \mapsto \min(a, b)$ and the multiplication $(a, b) \mapsto a + b$. Many of the classical algebraic constructions have interesting min-plus analogues. In particular, the characteristic polynomial function of a matrix $B \in \mathbb{R}_{\min}^{n\times n}$, already introduced by Cuninghame-Green [CG83], can be defined as the function which associates to a scalar x the permanent, in the min-plus sense, of the matrix $xI \oplus B$, where I is the min-plus identity matrix, "$\oplus$" denotes the min-plus addition, and the concatenation denotes the min-plus multiplication. The permanent, in the min-plus sense, of a matrix B, is the value of an optimal assignment in the weighted bipartite graph associated with B. A result of Cuninghame-Green and Meijer [CGM80] shows that a min-plus polynomial function $p(x)$ can be factored uniquely as $p(x) = a(x \oplus x_1)\cdots(x \oplus x_n)$, where $a, x_1, \ldots, x_n \in \mathbb{R}_{\min}$. The numbers $x_1, \ldots, x_n$, which coincide with the points of non-differentiability of p (counted with appropriate multiplicities), are called the *roots* or *corners* of p. The sequence of roots of the min-plus characteristic polynomial of a matrix $B \in \mathbb{R}_{\min}^{n\times n}$ can be computed in polynomial time, by solving $O(n)$ optimal assignment problems, as shown by Burkard and Butkovič [BB03]. The reader seeking information on the min-plus semiring may consult [CG79, MS92, BCOQ92, Max94, CG95, Gun98, KM97, GP97, Pin98, GM02, LM05].

We assume that $\mathcal{A}_\epsilon$ is given by (1). This allows one to handle the case of a perturbed matrix $\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b$, where the matrix b is non-generic.

The first main result of the present paper, Theorem 3.8, shows that the sequence of leading exponents of the eigenvalues of the matrix $\mathcal{A}_\epsilon$ is weakly (super) majorised by the sequence of roots of the min-plus characteristic polynomial of the matrix

of leading exponents of $\mathcal{A}_\epsilon$, and that the equality holds for generic values of the coefficients a_{ij}.

The proof of Theorem 3.8 relies on a variant of the Newton-Puiseux theorem in which the data are only assumed to have first order asymptotics, that we state as Theorem 3.1 in a way which illuminates the role of min-plus algebra. We consider the branches $\mathcal{Y}(\epsilon)$ solutions of the equation $\mathcal{P}(\epsilon, \mathcal{Y}(\epsilon)) = 0$, where $\mathcal{P}(\epsilon, Y) = \sum_{j=0}^{n} \mathcal{P}_j(\epsilon) Y^j$ and the $\mathcal{P}_j(\epsilon)$ are continuous functions, such that $\mathcal{P}_j(\epsilon) = p_j \epsilon^{P_j} + o(\epsilon^{P_j})$, with $p_j \in \mathbb{C}$ and $P_j \in \mathbb{R} \cup \{+\infty\}$. We characterise the cases where this information is enough to determine the first order asymptotics of the branches $\mathcal{Y}_1(\epsilon), \dots, \mathcal{Y}_n(\epsilon)$. Then, the leading exponents of the branches are precisely the roots of the min-plus polynomial $P(Y) = \bigoplus_{j=0}^{n} P_j Y^j$: the leading exponents of the classical roots are the min-plus roots. Note that in this case, by Legendre-Fenchel duality, the roots of the min-plus polynomial $P(Y)$ are precisely the slopes of the Newton-Polygon classically associated to $\mathcal{P}(\epsilon, Y)$. Note also that the generic equality in Theorem 3.8 could be derived from Murota's combinatorial relaxation technique [Mur90], which uses a parametric assignment problem, whose value is exactly the min-plus characteristic polynomial.

Theorem 3.8 determines the generic leading exponents of the eigenvalues of $\mathcal{A}_\epsilon$, but it does not determine the coefficients λ_i. To compute these coefficients, we define, in terms of eigenvalues of min-plus Schur complements, a sequence of *critical values* of A, that we characterise as generalised circuit means. We show that the sequence of roots of the min-plus characteristic polynomial of A is weakly majorised by the sequence of critical values of A (Theorem 4.6), and we characterise the equality case in terms of the existence of disjoint circuit covers, or perfect matchings, in certain graphs.

Our second main result, Theorem 5.1, shows that, in the equality case of Theorem 4.6, the coefficients λ_i can be obtained in terms of eigenvalues of certain Schur complements constructed from the matrix a. The theorem of Višik, Ljusternik and Lidskiĭ is a special case of this result (Corollary 7.1). We give in Section 7.3 examples of singular cases which can be solved by Theorem 5.1. In the remaining singular cases, different methods should be used, along the lines of [Mur90, ABG04].

We also prove an asymptotic result for eigenvectors, Theorem 6.1, which is analogous to Theorem 5.1. However, the combinatorial characterisation of the cases where Theorem 6.1 determines the generic asymptotics of all the entries of eigenvectors is lacking, see Section 6.3. Note that even when the first order asymptotics of an eigenvalue is determined, a detailed asymptotic information on $\mathcal{A}_\epsilon$ may be needed to determine the first order asymptotics of the corresponding eigenvector, as shown in our earlier work [ABG98] which concerns the special case of the Perron eigenvector.

The present results provide a new illustration of the role of min-plus algebra in asymptotic analysis, which was recognised by Maslov [Mas73, Ch. VIII]. He observed that WKB-type or large deviation type asymptotics lead to limiting equations, like Hamilton-Jacobi equations, satisfying some idempotent superposition principle. So, min-plus algebra arises as the limit of a deformation of usual algebra. This observation is at the origin of idempotent analysis [MS92, KM97, LMS01]. It has been used by Dobrokhotov, Kolokoltsov, and Maslov [DKM92, KM97] to obtain precise large deviation asymptotics concerning the Green kernel and the first eigenvalues of

a class of linear partial differential equations, with application to the Schrödinger equation.

The same deformation has been identified by Viro [Vir01], in relation with the patchworking method he developed for real algebraic curves. It appears in several recent works in "tropical geometry", in particular, by Mikhalkin [Mik01, Mik03], Forsberg, Passare, and Tsikh [FPT00], Passare and Rullgard [PR04], and Speyer and Sturmfels [SS04], following the introduction of amœbas of algebraic varieties by Gelfand, Kapranov, and Zelevinsky [GKZ94]. In these works, the relation between Newton polytopes and min-plus or "tropical" polynomials is apparent. We use the same relation in the version of the Newton-Puiseux theorem concerning first order asymptotics that we stated as Theorem 3.1.

Relations between max-plus algebra and asymptotic problems have also appeared in other contexts. Puhalski [Puh01] applied idempotent techniques to large deviations theory. Friedland [Fri86] observed that the max-plus eigenvalue can be obtained as a limit of the Perron root. Olsder and Roos [OR88] and De Schutter and De Moor [DSDM98] used asymptotics theorems to derive certain max-plus algebraic identities.

Finally, we note that Theorem 5.1 was announced in [ABG01], and that the role of min-plus roots in the Newton-Puiseux theorem was mentioned in [GP01].

2. Preliminaries

In this section, we recall some classical facts of min-plus algebra and show preliminary results. See for instance [BCOQ92] for more details.

The *min-plus semiring*, $\mathbb{R}_{\min}$, is the set $\mathbb{R}\cup\{+\infty\}$ equipped with the addition $(a,b)\mapsto a\oplus b=\min(a,b)$ and the multiplication $(a,b)\mapsto a\otimes b=a+b$. We shall denote by $\mathbb{0}=+\infty$ and $\mathbb{1}=0$ the zero and unit elements of $\mathbb{R}_{\min}$, respectively. We shall use the familiar algebraic conventions, in the min-plus context. For instance, if A,B are matrices of compatible dimensions with entries in $\mathbb{R}_{\min}$, $(AB)_{ij}=(A\otimes B)_{ij}=\bigoplus_k A_{ik}B_{kj}=\min_k(A_{ik}+B_{kj})$, $A^2=A\otimes A$, etc. Moreover, if $x\in\mathbb{R}_{\min}\setminus\{\mathbb{0}\}$, then x^{-1} is the inverse of x for the $\otimes$ law, that is $-x$, with the conventional notation. We shall also denote by $\overline{\mathbb{R}}_{\min}$ the complete min-plus semiring, which is the set $\mathbb{R}\cup\{\pm\infty\}$ equipped, as $\mathbb{R}_{\min}$, with the min and $+$ laws, with the convention $+\infty+(-\infty)=-\infty+(+\infty)=+\infty$.

2.1. Min-plus spectral theorem.

To any $n\times n$ matrix A with entries in a semiring S, we associate the directed graph $G(A)$, which has nodes $1,\dots,n$ and an arc (i,j) if $A_{ij}\neq\mathbb{0}$, where $\mathbb{0}$ denotes the zero element of S. We say that A is *irreducible* if $G(A)$ is strongly connected.

We next recall some results of min-plus spectral theory: the min-plus version of the Perron-Frobenius theorem has been discovered, rediscovered, precised or extended, by many authors [CG79, Vor67, Rom67, GM77, CDQV83, MS92]. Recent presentations can be found in [BCOQ92, CG95, GP97, Bap98, AGW].

Theorem 2.1 (Min-plus eigenvalue, see e.g. [BCOQ92, Th. 3.23])**.** *An irreducible matrix $A\in(\mathbb{R}_{\min})^{n\times n}$ has a unique eigenvalue:*

$$\rho_{\min}(A)=\bigoplus_{k=1}^{n}\bigoplus_{i_1,\dots,i_k}(A_{i_1i_2}\cdots A_{i_ki_1})^{\frac{1}{k}}\quad. \tag{2}$$

With the usual notations, (2) can be rewritten as

$$\rho_{\min}(A) = \min_{1\leq k\leq n} \min_{i_1,\dots,i_k} \frac{A_{i_1i_2} + \cdots + A_{i_ki_1}}{k} \quad .$$

If $p = (i_0, i_1, \dots, i_k)$ is a path of $G(A)$, we denote by $|p|_A = A_{i_0i_1} + \cdots + A_{i_{k-1}i_k}$ the *weight* of p, and by $|p| = k$ its *length*. Since any circuit of $G(A)$ can be decomposed in elementary circuits, which are of length at most n, $\rho_{\min}(A)$ is the *minimal circuit mean*:

$$\rho_{\min}(A) = \min_{c \text{ circuit in } G(A)} \frac{|c|_A}{|c|} \quad . \tag{3}$$

We say that a circuit $c = (i_1, i_2, \dots, i_k, i_1)$ of $G(A)$ is *critical* if c attains the minimum in (3), and we call critical the nodes and arcs of this circuit. The critical nodes and critical arcs form the *critical graph*, $G^c(A)$. We call *critical classes* the strongly connected components of $G^c(A)$. We will also use the name "critical class" for the set of nodes of a critical class.

The *Kleene's star* of a matrix $A \in \mathbb{R}_{\min}^{n\times n}$ is defined by

$$A^* = I \oplus A \oplus A^2 \oplus \cdots \in \overline{\mathbb{R}}_{\min}^{n\times n} \quad ,$$

i.e. $(A^*)_{ij} = \inf_{k\geq 0}(A^k)_{ij}$, where $I = A^0$ is the identity matrix (we shall use the same notation I for the identity matrix of $\mathbb{R}_{\min}^{n\times n}$, and for the identity matrix of $\mathbb{C}^{n\times n}$, for any n).

Proposition 2.2 (See e.g. [BCOQ92, Th. 3.20])**.** *All the entries of A^* are $> -\infty$ if, and only if, $\rho_{\min}(A) \geq 0$. Moreover, when $\rho_{\min}(A) \geq 0$,*

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1} \quad .$$

Theorem 2.3 (Min-plus eigenvectors, see e.g. [BCOQ92, Th. 3.100])**.** *Let $A \in \mathbb{R}_{\min}^{n\times n}$ be an irreducible matrix, and consider $\widetilde{A} = \rho_{\min}(A)^{-1}A$. Any eigenvector of A is a linear combination of the columns $\widetilde{A}^*_{\cdot,j}$ corresponding to critical nodes j. More precisely, if we select (arbitrarily) one node j per critical class and take the corresponding column $\widetilde{A}^*_{\cdot,j}$, we obtain a minimal generating set of the eigenspace of A.*

(In Theorem 2.3, and in the sequel, we write $\widetilde{A}^*_{\cdot,j}$ the j-th column of $(\widetilde{A})^*$.)

Given a matrix $A \in \mathbb{R}_{\min}^{n\times n}$ and a vector $V \in \mathbb{R}_{\min}^{n}$, we define the *saturation graph*, $\mathrm{Sat}(A, V)$, which has nodes $1, \dots, n$, and an arc (i, j) if $(AV)_i = A_{ij}V_j$ (that is $(AV)_i = A_{ij} + V_j$ with the usual notations). The following simple result relates the critical graph and the saturation graph.

Proposition 2.4 (See e.g. [BCOQ92, Th. 3.98])**.** *Let $A \in \mathbb{R}_{\min}^{n\times n}$ be an irreducible matrix with eigenvalue α, and let $V \in \mathbb{R}_{\min}^{n} \setminus \{\mathbb{0}\}$. If $AV = \alpha V$, then the strongly connected components of $\mathrm{Sat}(A, V)$ are exactly the strongly connected components of $G^c(A)$.*

In fact, Theorem 3.98 of [BCOQ92] only shows that any circuit of the saturation graph belongs to the critical graph, but the converse is straightforward.

The following elementary result is a special version of a maximum principle for ergodic control problems, see [AG03, Lemma 3.3] for more background, and [CTGG99, Lemma 1.4] for a proof in the min-plus case.

Proposition 2.5. *Let $A \in \mathbb{R}_{\min}^{n\times n}$ be an irreducible matrix with eigenvalue α, and let $V \in \mathbb{R}_{\min}^{n}$. If $AV \geq \alpha V$, then $(AV)_i = \alpha V_i$ for all critical nodes i of A.*

The saturation graphs associated to the generators of the eigenspace have a remarkable structure. Say that a strongly connected component C of a graph is *final* if for each node i, there is a path from i to C, and if there is no arc leaving C.

Proposition 2.6. *Let $A \in \mathbb{R}_{\min}^{n\times n}$ be an irreducible matrix with eigenvalue α, let $\widetilde{A} = \alpha^{-1}A$, let C be a critical class of A, and let V be an eigenvector of A. The following assertions are equivalent:*

1. *V is proportional to $\widetilde{A}^*_{\cdot j}$, for some $j \in C$;*
2. *C is the unique final class of* $\mathrm{Sat}(A, V)$.

Proof. We first prove $1 \implies 2$. It is enough to consider the case when $V = \widetilde{A}^*_{\cdot j}$. Since A is irreducible, all the entries of $\widetilde{A}^*$ are $< +\infty$. Moreover, since $\rho_{\min}(\widetilde{A}) = 0$, Proposition 2.2 yields $\widetilde{A}^* = I \oplus \widetilde{A} \oplus \cdots \oplus \widetilde{A}^{n-1}$. Hence, for all $i \neq j$, there exists a path $p = (i_0 = i, i_1, \ldots, i_k = j)$ from i to j, with length $1 \leq k \leq n-1$, and minimal weight, that is $\widetilde{A}^*_{ij} = \widetilde{A}_{i_0 i_1} \cdots \widetilde{A}_{i_{k-1} i_k}$. By Bellman's optimality principle, for all $0 \leq l \leq m \leq k$, the sub-path $(i_l, \ldots, i_m)$ has minimal weight: $\widetilde{A}^*_{i_l i_m} = \widetilde{A}_{i_l i_{l+1}} \cdots \widetilde{A}_{i_{m-1} i_m}$. Then, $\widetilde{A}^*_{i_l j} = \widetilde{A}_{i_l i_{l+1}} \widetilde{A}^*_{i_{l+1} j}$, that is, $\alpha V_{i_l} = A_{i_l i_{l+1}} V_{i_{l+1}}$, and $(i_l, i_{l+1}) \in \mathrm{Sat}(A, V)$ for all $l = 0, \ldots, k-1$. So for $i \neq j$, there is a path from i to j in $\mathrm{Sat}(A, V)$.

Assume, by contradiction, that there exists $k \in C$ and $l \notin C$ such that $(k, l) \in \mathrm{Sat}(A, V)$. Since $l \neq j$, there is a path from l to j in $\mathrm{Sat}(A, V)$, and since C is a strongly connected component of $\mathrm{Sat}(A, V)$ (by Proposition 2.4), there is a path from j to k in $\mathrm{Sat}(A, V)$, which yields a circuit of $\mathrm{Sat}(A, V)$ passing through C and $k \notin C$. This contradicts the fact that C is a strongly connected component of $\mathrm{Sat}(A, V)$.

We finally prove $2 \implies 1$. Assume that C is the unique final class of $\mathrm{Sat}(A, V)$, and let us fix $j \in C$. Then, for each i, we can find a path $(i_0 = i, \ldots, i_k = j)$ from i to j in $\mathrm{Sat}(A, V)$, so that $V_{i_0} = \widetilde{A}^*_{i_0 i_1} V_{i_1}, \ldots, V_{i_{k-1}} = \widetilde{A}^*_{i_{k-1} i_k} V_{i_k}$. Hence, $V_i = \widetilde{A}^*_{i_0 i_1} \cdots \widetilde{A}^*_{i_{k-1} i_k} V_j \leq \widetilde{A}^*_{ij} V_j$. The other inequality holds, since $V = \widetilde{A} V$ implies $V = \widetilde{A}^* V$. Thus, $V = \widetilde{A}^*_{\cdot j} V_j$ is proportional to $\widetilde{A}^*_{\cdot j}$. □

2.2. **Min-plus polynomials.** We recall here some results about formal polynomials and polynomial functions over $\mathbb{R}_{\min}$, and in particular a min-plus analogue of “the fundamental theorem of algebra”, which is due to Cuninghame-Green and Meijer [CGM80]. The connection between the min-plus evaluation morphism and the Fenchel transform, was already observed in [CGNQ89] and [BCOQ92, Section 3.3.1].

We denote by $\mathbb{R}_{\min}[\mathsf{Y}]$ the semiring of formal polynomials with coefficients in $\mathbb{R}_{\min}$ in the indeterminate Y: a *formal polynomial* $P \in \mathbb{R}_{\min}[\mathsf{Y}]$ is nothing but a sequence $(P_k)_{k\in\mathbb{N}} \in \mathbb{R}_{\min}^{\mathbb{N}}$ such that $P_k = \mathbb{0}$ for all but finitely many values of k. Formal polynomials are equipped with the entry-wise sum, $(P \oplus Q)_k = P_k \oplus Q_k$, and the Cauchy product, $(PQ)_k = \bigoplus_{0\leq i\leq k} P_i Q_{k-i}$. As usual, we denote a formal polynomial P as a formal sum, $P = \bigoplus_{k=0}^{\infty} P_k \mathsf{Y}^k$. We also define the *degree* and *valuation* of P: $\deg P = \sup\{k \in \mathbb{N} \mid P_k \neq \mathbb{0}\}$, $\operatorname{val} P = \inf\{k \in \mathbb{N} \mid P_k \neq \mathbb{0}\}$ ($\deg P = -\infty$ and $\operatorname{val} P = +\infty$ if $P = \mathbb{0}$). To any $P \in \mathbb{R}_{\min}[\mathsf{Y}]$, we associate the

polynomial function $\widehat{P}:\mathbb{R}_{\min}\to\mathbb{R}_{\min},\ y\mapsto \widehat{P}(y)=\bigoplus_{k=0}^{\infty} P_k y^k$, that is, with the usual notation:

$$\widehat{P}(y)=\min_{k\in\mathbb{N}}(P_k+ky) \enspace . \tag{4}$$

We denote by $\mathbb{R}_{\min}\{\mathsf{Y}\}$ the semiring of polynomial functions $\widehat{P}$. Contrary to the case of real or complex polynomials, the evaluation morphism, $\mathbb{R}_{\min}[\mathsf{Y}]\to\mathbb{R}_{\min}\{\mathsf{Y}\},\ P\mapsto\widehat{P}$ is not injective. Indeed, the evaluation morphism is essentially a specialisation of the Fenchel transform over $\mathbb{R}$:

$$\mathcal{F}:\overline{\mathbb{R}}^{\mathbb{R}}\to\overline{\mathbb{R}}^{\mathbb{R}},\ \mathcal{F}(f)(y)=\sup_{x\in\mathbb{R}}(xy-f(x)) \enspace ,$$

since, for all $y\in\mathbb{R}$, $\widehat{P}(y)=-\mathcal{F}(P)(-y)$, where P is extended to a function

$$P:\mathbb{R}\to\overline{\mathbb{R}},\ x\mapsto P(x),\ \text{with } P(x)=\begin{cases}P_k & \text{if } x=k\in\mathbb{N}\\ +\infty & \text{otherwise}\end{cases} \enspace , \tag{5}$$

It follows from (4) that $\widehat{P}$ is a concave nondecreasing function with integer slopes.

In the sequel, we denote by $\operatorname{vex} f$ the convex hull of a map $f:\mathbb{R}\to\overline{\mathbb{R}}$, and we denote by $\overline{P}$ the formal polynomial whose sequence of coefficients is obtained by restricting to $\mathbb{N}$ the convex hull of the map $P:\mathbb{R}\to\overline{\mathbb{R}}$. Thus, $\overline{P}_k=(\operatorname{vex} P)(k)$. The following result is a special case of the Legendre-Fenchel inversion theorem [Roc70, Section 12].

Proposition 2.7. *If $P\in\mathbb{R}_{\min}[\mathsf{Y}]$, then $\overline{P}$ is the minimal formal polynomial Q such that $\widehat{Q}=\widehat{P}$, we have $\overline{\overline{P}}=\overline{P}$, and $\overline{P}$ is given by*

$$\overline{P}_k=\sup_{y\in\mathbb{R}}(-ky+\widehat{P}(y)) \enspace .$$

Theorem 2.8 ([BCOQ92, Th. 3.43, 1 and 2])**.** *A formal polynomial of degree n, $P\in\mathbb{R}_{\min}[\mathsf{Y}]$, satisfies $P=\overline{P}$ if, and only if, there exist $c_1\leq\cdots\leq c_n\in\mathbb{R}_{\min}$ such that*

$$P=P_n(\mathsf{Y}\oplus c_1)\cdots(\mathsf{Y}\oplus c_n) \enspace .$$

The c_i are unique and given, by:

$$c_i=\begin{cases}P_{n-i}(P_{n-i+1})^{-1} & \text{if } P_{n-i+1}\neq \mathbb{0}\\ \mathbb{0} & \text{otherwise,}\end{cases} \qquad \text{for } i=1,\ldots,n \enspace . \tag{6}$$

The min-plus analogue of the fundamental theorem of algebra due to Cuninghame-Green and Meijer can be obtained by applying Theorem 2.8 to $\overline{P}$, since $\overline{\overline{P}}=\overline{P}$ and $\widehat{\overline{P}}=\widehat{P}$.

Theorem 2.9 ([CGM80])**.** *Any polynomial function $\widehat{P}\in\mathbb{R}_{\min}\{\mathsf{Y}\}$ can be factored in a unique way as*

$$\widehat{P}(y)=P_n(y\oplus c_1)\cdots(y\oplus c_n) \enspace , \tag{7}$$

with $c_1\leq\cdots\leq c_n$.

The c_i are called the *roots* of $\widehat{P}$. (In [CGM80], the term *corners* is used as a synonym of root, we use the term of root which makes the analogy with classical algebra clearer.) The *multiplicity* of the root c is the cardinality of the set $\{j\in\{1,\ldots,n\}\mid c_j=c\}$. We shall denote by $\mathsf{R}(\widehat{P})$ the sequence of roots:

$\mathsf{R}(\widehat{P}) = (c_1, \ldots, c_n)$. By extension, if $P \in \mathbb{R}_{\min}[\mathsf{Y}]$ is a formal polynomial, we will call *roots* of P the roots of $\widehat{P}$: $\mathsf{R}(P) = \mathsf{R}(\widehat{P})$. By Proposition 2.7, $\mathsf{R}(P) = \mathsf{R}(\overline{P})$. Geometrically, the function $\overline{P}$ is the restriction to $\mathbb{N}$ of the convex function $\operatorname{vex} P$, which is piecewise affine on its support, $[\operatorname{val} P, \deg P]$, and $\widehat{P}$ is concave, piecewise affine.

Proposition 2.10. *The roots $c \in \mathbb{R}$ of a formal polynomial $P \in \mathbb{R}_{\min}[\mathsf{Y}]$ are exactly the points at which $\widehat{P}$ is not differentiable. They coincide with the opposites of the slopes of the affine parts of $\operatorname{vex} P : [\operatorname{val} P, \deg P] \to \mathbb{R}$. The multiplicity of a root $c \in \mathbb{R}$ is equal to the variation of slope of $\widehat{P}$ at c, $\widehat{P}'(c^-) - \widehat{P}'(c^+)$, and it coincides with the length of the interval where $\operatorname{vex} P$ has slope $-c$. Moreover, $\mathbb{0}$ is a root of P if, and only if, $\widehat{P}'(\mathbb{0}^-) := \lim_{c \to +\infty} \widehat{P}'(c) \neq 0$. In that case $\widehat{P}'(\mathbb{0}^-)$ is the multiplicity of $\mathbb{0}$, and it coincides with $\operatorname{val} P$.*

Proof. The characterisation of the roots and of their multiplicities in terms of $\widehat{P}$ is due to Cuninghame-Green and Meijer [CGM80]. It can be deduced from (7), since when $c \in \mathbb{R}$, $\widehat{Q}(y) := (y \oplus c)^k = k \min(y, c)$ has c as unique point of non differentiability, with $\widehat{Q}'(c^-) = k$ and $\widehat{Q}'(c^+) = 0$. The case where $c = \mathbb{0}$ is a straightforward consequence of (7). The characterisation of the roots and of their multiplicities in terms of $\operatorname{vex} P$ follows from (6), since when $c_i \in \mathbb{R}$, $c_i = \overline{P}_{n-i} - \overline{P}_{n-i+1} = (\operatorname{vex} P)'(x)$ for all $x \in (n-i, n-i+1)$, and $c_i = \mathbb{0} \implies \overline{P}_{n-i} = \overline{P}_n c_1 \cdots c_i = \mathbb{0}$. □

The duality between roots and slopes in Proposition 2.10 is a special case of the Legendre-Fenchel duality formula for subdifferentials: $-c \in \partial(\operatorname{vex} P)(x) \Leftrightarrow x \in \partial \mathcal{F}(P)(-c) \Leftrightarrow x \in \partial^+ \widehat{P}(c)$ where ∂ and ∂^+ denote the subdifferential and superdifferential, respectively [Roc70, Th. 23.5].

Lemma 2.11. *Let $P = \bigoplus_{i=0}^n P_i \mathsf{Y}^i \in \mathbb{R}_{\min}[\mathsf{Y}]$ be a formal polynomial of degree n. Then, $\mathsf{R}(P) = (c_1 \leq \cdots \leq c_n)$ if, and only if, $P \geq P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ and*

$$P_{n-i} = P_n c_1 \cdots c_i \quad \text{for all } i \in \{0, n\} \cup \{i \in \{1, \ldots, n-1\} \mid c_i < c_{i+1}\} \ . \tag{8}$$

In particular, $P_{n-i} = \overline{P}_{n-i}$ holds for all i as in (8).

Proof. We first prove the "only if" part. If $\mathsf{R}(P) = (c_1 \leq \cdots \leq c_n)$, then $\overline{P} = \overline{P}_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ and $\overline{P}_{n-i} = \overline{P}_n c_1 \cdots c_i$ for all $i = 1, \ldots n$. Recall that P defines a map $x \mapsto P(x)$ by (5). By definition of $\operatorname{vex} P$, the epigraph of $\operatorname{vex} P$, $\operatorname{epi} \operatorname{vex} P$, is the convex hull of the epigraph of P, $\operatorname{epi} P$. By a classical result [Roc70, Cor 18.3.1], if S is a set with convex hull C, any extreme point of C belongs to S. Let us apply this to $S = \operatorname{epi} P$ and $C = \operatorname{epi} \operatorname{vex} P$. Since $\overline{P}_{n-i} = \overline{P}_n c_1 \cdots c_i$, the piecewise affine map $\operatorname{vex} P$ changes its slope at any point $n-i$ such that $c_i < c_{i+1}$. Thus, any point $(n-i, \operatorname{vex} P(n-i))$ with $c_i < c_{i+1}$ is an extreme point of $\operatorname{epi} \operatorname{vex} P$, which implies that $(n-i, \operatorname{vex} P(n-i)) \in \operatorname{epi} P$, i.e., $P_{n-i} \leq \operatorname{vex} P(n-i) = \overline{P}_{n-i}$. Since the other inequality is trivial by definition of the convex hull, we have $P_{n-i} = \overline{P}_{n-i}$. Obviously, P and $\overline{P}$ have the same degree, which is equal to n, and they have the same valuation, k. Then, $(n, \operatorname{vex} P(n))$ and $(k, \operatorname{vex} P(k))$ are extreme points of $\operatorname{epi} \operatorname{vex} P$, and by the preceding argument, $P_n = \overline{P}_n$, and $P_k = \overline{P}_k$. Hence, $P_0 = \overline{P}_0$, if $k = 0$, and $P_0 = \overline{P}_0 = +\infty$, if $k > 0$. We have shown (8), together with the last statement of the lemma. Since $\overline{P}_n = P_n$ and $P \geq \overline{P}$, we also obtain $P \geq P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$.

For the "if" part, assume that $P \geq P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ and that (8) holds. Since $Q = P_n(\mathsf{Y} \oplus c_1) \cdots (\mathsf{Y} \oplus c_n)$ is convex, and the convex hull map $P \mapsto \overline{P}$ is monotone, we must have $\overline{P} \geq \overline{Q} = Q$. Hence, $P \geq \overline{P} \geq Q$ and since $P_{n-i} = Q_{n-i}$ for all i as in (8), we must have $\overline{P}_{n-i} = Q_{n-i}$, thus $\operatorname{vex} P(n-i) = \operatorname{vex} Q(n-i)$ at these i. Since $\operatorname{vex} P$ is convex, since $\operatorname{vex} Q$ is piecewise affine and $\operatorname{vex} Q(j) = \operatorname{vex} P(j)$ for j at the boundary of the domain of $\operatorname{vex} Q$ and at all the j where $\operatorname{vex} Q$ changes of slope, we must have $\operatorname{vex} P = \operatorname{vex} Q$. Hence $\overline{P} = \overline{Q} = Q$ and $\mathsf{R}(P) = \mathsf{R}(\overline{P}) = \mathsf{R}(Q) = (c_1, \ldots, c_n)$. $\square$

The above notions are illustrated in Figure 1, where we consider the formal min-plus polynomial $P = \mathsf{Y}^3 \oplus 5\mathsf{Y}^2 \oplus 6\mathsf{Y} \oplus 13$. The map $j \mapsto P_j$, together with the map $\operatorname{vex} P$, are depicted at the left of the figure, whereas the polynomial function $\widehat{P}$ is depicted at the right of the figure. We have $\overline{P} = \mathsf{Y}^3 \oplus 3\mathsf{Y}^2 \oplus 6\mathsf{Y} \oplus 13 = (\mathsf{Y} \oplus 3)^2(\mathsf{Y} \oplus 7)$. Thus, the roots of P are 3 and 7, with respective multiplicities 2 and 1. The roots are visualised at the right of the figure, or alternatively, as the opposite of the slopes of the two line segments at the left of the figure. The multiplicities can be read either on the map $\widehat{P}$ at the right of the figure (the variation of slope of $\widehat{P}$ at points 3 and 7 is 2 and 1, respectively), or on the map $\operatorname{vex} P$ at the left of the figure (as the respective horizontal widths of the two segments).

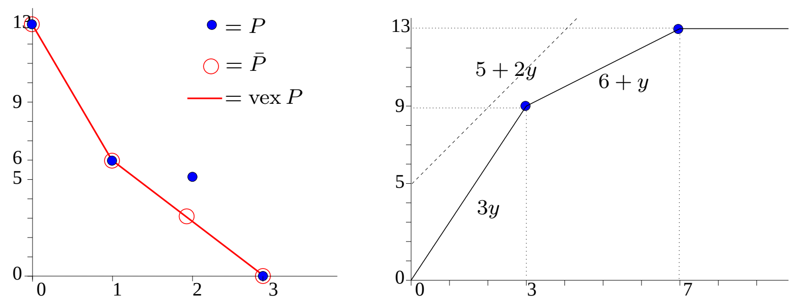


FIGURE 1. The formal min-plus polynomial $P = \mathsf{Y}^3 \oplus 5\mathsf{Y}^2 \oplus 6\mathsf{Y} \oplus 13$ and its associated polynomial function $\widehat{P}$.

2.3. **Schur complements.** We recall here the definitions of conventional and min-plus Schur complements. We shall consider matrices indexed by "abstract indices": if L and M are finite sets and S is a semiring, a $L \times M$ matrix with values in S is an element A of $S^{L \times M}$ and the entries of A are denoted by A_{ij} with $i \in L$ and $j \in M$. Moreover, for all $J \subset L$ and $K \subset M$, we denote by A_{JK} the $J \times K$ submatrix of A: $A_{JK} = (A_{jk})_{j \in J,\, k \in K}$. This definition applies to $n \times n$ matrices by taking $L = M = \{1, \ldots, n\}$. Graphs of $L \times L$ matrices A are defined as for $n \times n$ matrices (see Section 2.1) with the only difference that the set of nodes is L.

Definition 2.12. Let $C \subset L$ be finite sets, and let $N = L \setminus C$. If a is a $L \times L$ matrix with entries in $\mathbb{C}$, and if a_{CC} is invertible, the *Schur complement* of C in a is defined by

$$\operatorname{Schur}(C, a) = a_{NN} - a_{NC}(a_{CC})^{-1} a_{CN} \quad .$$

Definition 2.13. Let $C \subset L$ be finite sets, and let $N = L \setminus C$. If A is a $L \times L$ matrix with entries in $\mathbb{R}_{\min}$, $\lambda \in \mathbb{R}_{\min} \setminus \{\mathbb{0}\}$, and $\rho_{\min}(\lambda^{-1}A_{CC}) \geq 0$, the min-plus λ-*Schur complement* of C in A is defined by

$$\mathrm{Schur}(C, \lambda, A) = A_{NN} \oplus A_{NC}(\lambda^{-1}A_{CC})^* \lambda^{-1} A_{CN} \quad . \tag{9}$$

When $\lambda = \mathbb{1} = 0$, we shall simply write $\mathrm{Schur}(C, A)$ instead of $\mathrm{Schur}(C, \mathbb{1}, A)$.

In fact, in the sequel, we shall mostly use min-plus Schur complement corresponding to $\lambda = \rho_{\min}(A)$. The goal of the insertion of the normalising factors in (9) is to get the following homogeneity property:

$$\mathrm{Schur}(C, \mu\lambda, \mu A) = \mu \, \mathrm{Schur}(C, \lambda, A) \quad , \tag{10}$$

for all $\lambda, \mu \in \mathbb{R}$ such that $\lambda \leq \rho_{\min}(A_{CC})$ and $\mu\lambda \leq \rho_{\min}(A_{CC})$.

Using the same symbol, "Schur", both for conventional and min-plus Schur complements is not ambiguous: considering min-plus Schur complements of complex matrices, or conventional Schur complements of min-plus matrices, would be meaningless.

Both min-plus and conventional Schur complements satisfy

$$\mathrm{Schur}(C \cup C', a) = \mathrm{Schur}(C, \mathrm{Schur}(C', a)) \tag{11}$$

for all $L \times L$ matrices a, and for all disjoint subsets of indices $C, C' \subset L$, provided that the Schur complements are well defined (if $\mathrm{Schur}(C', a)$ is well defined, then the left hand side of (11) exists if, and only if, its right hand side exists). Of course, (11) is a classical Gaussian elimination identity, which is well known, both in conventional algebra and in the min-plus algebra (the left hand side and the right hand side of (11) are unambiguous rational expressions, with elementary interpretations in terms of paths, see for instance [Lal79] for more background).

Finally, if $K \subset L$ and if b is the $K \times K$ submatrix of a, we shall sometimes write abusively $\mathrm{Schur}(b, a)$, instead of $\mathrm{Schur}(K, a)$.

We now give some graph interpretations of the weights and eigenvalues of min-plus Schur complements. Let G be a graph with set of nodes L, let C be a subset of L and set $N = L \setminus C$. For all paths $p = (i_0, \dots, i_k)$ of G, we denote by $|p|_C$ the number of arcs of p with initial node in C, i.e., $|p|_C = \#\{0 \leq m \leq k-1 \mid i_m \in C\}$, where $\#$ denotes the cardinality of a set. (All the path interpretations below have dual versions, obtained by replacing "initial" by "final".) We also denote by $p \cap C$ the subsequence of p obtained by deleting the nodes not in C ($p \cap C$ need not be a path of G). The following classical interpretation of Schur complements is an immediate consequence of the graph interpretation of the star.

Lemma 2.14. *Let $C \subset L$ be finite sets, and let $N = L \setminus C$. Let A be a $L \times L$ matrix with entries in $\mathbb{R}_{\min}$, and $\lambda \in \mathbb{R}_{\min} \setminus \{\mathbb{0}\}$ be such that $\rho_{\min}(A_{CC}) \geq \lambda$. Then, p is a path in $G(\mathrm{Schur}(C, \lambda, A))$ if, and only if, there exists a path p' in $G(A)$ with the same extremal nodes as p and such that $p' \cap N = p$. Moreover, for all paths p in $G(\mathrm{Schur}(C, \lambda, A))$, we have*

$$|p|_{\mathrm{Schur}(C,\lambda,A)} = \min |p'|_A - \lambda |p'|_C \quad ,$$

where the minimum is taken over all the paths p' of $G(A)$ that have the same extremal nodes as p and satisfy $p' \cap N = p$. In particular, c is a circuit in $G(\mathrm{Schur}(C, \lambda, A))$ if, and only if, there exists a circuit c' in $G(A)$ such that

$c' \cap N = c$; and for all circuits c in $G(\mathrm{Schur}(C, \lambda, A))$, we have

$$|c|_{\mathrm{Schur}(C,\lambda,A)} = \min |c'|_A - \lambda |c'|_C \enspace ,$$

where the minimum is taken over all the circuits c' of $G(A)$ such that $c' \cap N = c$.

Proposition 2.15. *Let $C \subset L$ be finite sets, and let $N = L \setminus C$. Let A be a $L \times L$ matrix with entries in $\mathbb{R}_{\min}$, and $\lambda \in \mathbb{R}_{\min} \setminus \{\mathbb{0}\}$ be such that $\rho_{\min}(A_{CC}) \geq \lambda$. Then,*

$$\rho_{\min}(\mathrm{Schur}(C, \lambda, A)) = \min \frac{|c'|_A - \lambda |c'|_C}{|c'| - |c'|_C} \tag{12}$$

where the minimum is taken over all the circuits c' of $G(A)$ which are not included in C. Moreover, c is a critical circuit of $\mathrm{Schur}(C, \lambda, A)$ if, and only if, there exists a circuit c' of $G(A)$ such that $c' \cap N = c$ and c' minimises (12).

Proof. Using (3) and Lemma 2.14, we get

$$\begin{aligned}
\rho_{\min}(\mathrm{Schur}(C, \lambda, A)) &= \min_{c \text{ circuit in } N} \frac{|c|_{\mathrm{Schur}(C,\lambda,A)}}{|c|} \\
&= \min_{c \text{ circuit in } N} \left(\min_{c' \text{ circuit of } G(A),\, c' \cap N = c} \frac{|c'|_A - \lambda |c'|_C}{|c|} \right) \\
&= \min_{c' \text{ circuit of } G(A),\, c' \cap N \neq \varnothing} \frac{|c'|_A - \lambda |c'|_C}{|c'| - |c'|_C} \enspace ,
\end{aligned}$$

since $|c' \cap N| = |c'| - |c'|_C$ for all circuits c'. This yields (12). If c is a critical circuit of $\mathrm{Schur}(C, \lambda, A)$, then $\rho_{\min}(\mathrm{Schur}(C, \lambda, A)) = (|c|_{\mathrm{Schur}(C,\lambda,A)})/|c|$ and by Lemma 2.14, there exists a circuit c' of $G(A)$ such that $c' \cap N = c$ and $|c|_{\mathrm{Schur}(C,\lambda,A)} = |c'|_A - \lambda |c'|_C$. Since in that case, $|c| = |c' \cap N| = |c'| - |c'|_C$, we deduce that c' minimises (12). Conversely, if c' minimises (12), then, $c = c' \cap N$ is nonempty and by Lemma 2.14, c is a circuit of $G(\mathrm{Schur}(C, \lambda, A))$. Moreover, by Lemma 2.14 again,

$$\rho_{\min}(\mathrm{Schur}(C, \lambda, A)) \leq \frac{|c|_{\mathrm{Schur}(C,\lambda,A)}}{|c|} \leq \frac{|c'|_A - \lambda |c'|_C}{|c'| - |c'|_C} = \rho_{\min}(\mathrm{Schur}(C, \lambda, A)) \enspace ,$$

thus c is a critical circuit of $\mathrm{Schur}(C, \lambda, A)$. □

Note that if c' is a circuit in C, that is if the denominator in (12) is zero, the numerator is necessarily nonnegative, since $\lambda \leq \rho_{\min}(A_{CC})$.

3. Min-plus polynomials, Newton-Puiseux theorem and generic exponents of eigenvalues

3.1. **Preliminaries on exponents and general assumptions.** Let $\mathcal{C}$ denote the set of continuous functions f from some interval $(0, \epsilon_0)$ to $\mathbb{C}$ with $\epsilon_0 > 0$, such that $|f(\epsilon)| \leq \epsilon^{-k}$ on $(0, \epsilon_0)$, for some positive constant k. Since all the properties that we will prove in the sequel will hold on some neighbourhoods of 0, we shall rather use the ring of *germs* at 0 of elements of $\mathcal{C}$, which is obtained by quotienting $\mathcal{C}$ by the equivalence relation that identifies functions which coincide on a neighbourhood of 0. This ring of germs will be also denoted by $\mathcal{C}$. For any germ $f \in \mathcal{C}$, we shall abusively denote by $f(\epsilon)$ or f_ϵ the value at ϵ of any representative of the germ f.

We shall make a similar abuse for vectors, matrices, polynomials whose coefficients are germs. We call *exponent* of $f \in \mathcal{C}$:

$$\mathsf{e}(f) \stackrel{\text{def}}{=} \liminf_{\epsilon \to 0} \frac{\log |f(\epsilon)|}{\log \epsilon} \in \mathbb{R} \cup \{+\infty\} \ . \tag{13}$$

We have, for all $f, g \in \mathcal{C}$ and $\lambda \in \mathbb{C}$,

$$\mathsf{e}(f+g) \geq \min(\mathsf{e}(f), \mathsf{e}(g)) \ , \tag{14}$$

$$\mathsf{e}(fg) \geq \mathsf{e}(f) + \mathsf{e}(g) \ , \tag{15}$$

with equality in (14) if $\mathsf{e}(f) \neq \mathsf{e}(g)$ and equality in (15) if the liminf in the definition of $\mathsf{e}(f)$ or $\mathsf{e}(g)$ is a limit. Thus, $f \mapsto \mathsf{e}(f)$ is "almost" a morphism $\mathcal{C} \to \mathbb{R}_{\min}$. In the sequel exponents will be considered as elements of $\mathbb{R}_{\min}$, so that (15) will be written as $\mathsf{e}(fg) \geq \mathsf{e}(f)\mathsf{e}(g)$. An element $f \in \mathcal{C}$ is invertible if, and only if, $\mathsf{e}(f) \neq \mathbb{0}$ (or equivalently, if there exists a positive constant such that $|f(\epsilon)| \geq \epsilon^k$). If f is invertible, its inverse is the map $f^{-1} : \epsilon \mapsto f(\epsilon)^{-1}$ and we have $\mathsf{e}(f^{-1}) \leq \mathsf{e}(f)^{-1}$ with equality if, and only if, the liminf in the definition of $\mathsf{e}(f)$ is a limit.

We shall say that $f \in \mathcal{C}$ has a *first order asymptotics* if

$$f(\epsilon) \sim a\epsilon^A, \qquad \text{when } \epsilon \to 0^+ \ , \tag{16}$$

with either $A \in \mathbb{R}$ and $a \in \mathbb{C} \setminus \{0\}$, or $A = +\infty$ and $a \in \mathbb{C}$. In the first case, (16) means that $\lim_{\epsilon \to 0} \epsilon^{-A} f(\epsilon) = a$, in the second case, (16) means that $f = 0$ (in a neighbourhood of 0). We have:

$$f(\epsilon) \sim a\epsilon^A \implies \mathsf{e}(f) = A \ , \tag{17}$$

and the liminf in (13) is a limit. We shall also need an equivalence notion slightly weaker than $\sim$. If $f \in \mathcal{C}$, $a \in \mathbb{C}$ and $A \in \mathbb{R}_{\min}$, we write

$$f(\epsilon) \simeq a\epsilon^A \tag{18}$$

if $f(\epsilon) = a\epsilon^A + o(\epsilon^A)$. If $A \in \mathbb{R}$, this means that $\lim_{\epsilon \to 0} \epsilon^{-A} f(\epsilon) = a$. If $A = +\infty$, this means by convention that $f = 0$. If $a \neq 0$ or $A = +\infty$, then $f(\epsilon) \simeq a\epsilon^A$ if, and only if, $f(\epsilon) \sim a\epsilon^A$ and in that case $\mathsf{e}(f) = A$. In general,

$$f(\epsilon) \simeq a\epsilon^A \implies \mathsf{e}(f) \geq A \ . \tag{19}$$

Conversely, $\mathsf{e}(f) > A \implies f(\epsilon) \simeq 0\epsilon^A$. Of course, in (18), $a\epsilon^A$ must be viewed as a formal expression, for the equivalence to be meaningful when $a = 0$ and $A \in \mathbb{R}$. In (17), however, $a\epsilon^A$ can be viewed either as a formal expression or as an element of $\mathcal{C}$.

Throughout the paper, we consider a matrix $\mathcal{A} \in \mathcal{C}^{n \times n}$ and we shall assume that the entries $(\mathcal{A}_\epsilon)_{ij}$ of $\mathcal{A}_\epsilon$ have asymptotics of the form:

$$\begin{aligned} (\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}, \ &\text{for some matrix } a = (a_{ij}) \in \mathbb{C}^{n \times n}, \\ &\text{and for some irreducible matrix } A = (A_{ij}) \in \mathbb{R}_{\min}^{n \times n}. \end{aligned} \tag{20}$$

(The case where A is reducible is a straightforward extension.) Under rather general circumstances (see Section 3.2), the eigenvalues $\mathcal{L}_\epsilon^1, \ldots, \mathcal{L}_\epsilon^n$ of $\mathcal{A}_\epsilon$ belong to $\mathcal{C}$ and have first order asymptotics:

$$\mathcal{L}_\epsilon^i \sim \lambda_i \epsilon^{\Lambda_i} \ . \tag{21}$$

We next relate the sequence $(\Lambda_1, \ldots, \Lambda_n)$ with two sequences constructed by using only the information on the exponents of the entries $(\mathcal{A}_\epsilon)_{ij}$ of the matrix $\mathcal{A}_\epsilon$ given by the A_{ij}.

3.2. **First order Newton-Puiseux theorem and min-plus polynomials.** The usual way to compute the Λ_i in (21) is to use the classical Newton-Puiseux theorem. We state here a general first order version of this theorem in a way which illuminates the role of min-plus algebra.

For any formal polynomial with coefficients in $\mathcal{C}$, $\mathcal{P}(\epsilon,\mathsf{Y})=\sum_{j=0}^{n}\mathcal{P}_j(\epsilon)\mathsf{Y}^j\in\mathcal{C}[\mathsf{Y}]$, we define the min-plus *polynomial of exponents*:

$$\mathsf{e}(\mathcal{P})\stackrel{\text{def}}{=}\bigoplus_{j=0}^{n}\mathsf{e}(\mathcal{P}_j)\mathsf{Y}^j\in\mathbb{R}_{\min}[\mathsf{Y}]\enspace .$$

The transformation of ordinary polynomials to min-plus (or "tropical") polynomial by the map e is instrumental in works on amoebas (for instance, a very similar definition is given in [SS04]).

Recall that to $P=\mathsf{e}(\mathcal{P})$ is associated the polynomial function $\widehat{P}$ and the convex formal polynomial $\overline{P}$, as in Section 2.2. For instance, to $\mathcal{P}=\mathsf{Y}^3+\epsilon^5\mathsf{Y}^2-\epsilon^6\mathsf{Y}+\epsilon^{13}$ corresponds the formal min-plus polynomial $P=\mathsf{e}(\mathcal{P})=\mathsf{Y}^3\oplus 5\mathsf{Y}^2\oplus 6\mathsf{Y}+13$ represented in Figure 1.

Theorem 3.1 (First order Newton-Puiseux theorem)**.** *Let $\mathcal{P}=\sum_{j=0}^{n}\mathcal{P}_j(\epsilon)\mathsf{Y}^j\in\mathcal{C}[\mathsf{Y}]$ such that $\mathcal{P}_n=1$. The following assertions are equivalent:*

1. *There exist $\mathcal{Y}_1,\dots,\mathcal{Y}_n\in\mathcal{C}$ such that $\mathcal{Y}_1(\epsilon),\dots,\mathcal{Y}_n(\epsilon)$ are the roots of $\mathcal{P}(\epsilon,y)=0$ counted with multiplicities, and $\mathcal{Y}_1,\dots,\mathcal{Y}_n$ have first order asymptotics, $\mathcal{Y}_j(\epsilon)\sim y_j\epsilon^{Y_j}$ with $Y_1\leq\cdots\leq Y_n$;*
2. *There exist $p=\sum_{j=0}^{n}p_j\mathsf{Y}^j\in\mathbb{C}[\mathsf{Y}]$ and $P=\bigoplus_{j=0}^{n}P_j\mathsf{Y}^j\in\mathbb{R}_{\min}[\mathsf{Y}]$ satisfying $\mathcal{P}_j(\epsilon)\simeq p_j\epsilon^{P_j}$, $j=0,\dots,n$, with $p_n=1$, $P_n=\mathbb{1}$, $p_0\neq 0$ or $P_0=\mathbb{0}$, and $p_{n-i}\neq 0$ for all $i\in\{1,\dots,n-1\}$ such that $c_i<c_{i+1}$, where $(c_1\leq\cdots\leq c_n)=\mathsf{R}(P)$.*

When these assertions hold, we have $\mathsf{e}(\mathcal{P})\geq P$, $\overline{\mathsf{e}(\mathcal{P})}=\overline{P}$, and $\mathsf{R}(\mathsf{e}(\mathcal{P}))=\mathsf{R}(P)=(c_1\leq\cdots\leq c_n)=(Y_1\leq\cdots\leq Y_n)$. Moreover, if $c\in\mathbb{R}$ is a root of P with multiplicity k and $c_i=\cdots=c_{i+k-1}=c$, then $y_i,\dots,y_{i+k-1}$ are precisely the non-zero roots of the polynomial

$$p^{(i)}=\sum_{\substack{0\leq j\leq n\\ \widehat{P}(c)=P_jc^j}}p_j\mathsf{Y}^j\in\mathbb{C}[\mathsf{Y}]\enspace ,$$

counted with multiplicities.

The classical Newton-Puiseux theorem applies to the case where $\mathcal{C}$ is replaced by the field of (formal, or convergent) Puiseux series (a Puiseux series is of the form $\sum_{k=K}^{\infty}a_kx^{k/s}$ with $a_k\in\mathbb{C}$, $K\in\mathbb{Z}$ and $s\in\mathbb{N}\setminus\{0\}$), and shows $2\Longrightarrow 1$ only. In the classical statement of the theorem, the leading exponents Y_i, are, up to an inversion and change of sign, the slopes of the Newton polygon, and the polynomials $p^{(i)}$ are defined in terms of the edges of the polygon. Since, when $P=\mathsf{e}(\mathcal{P})$, the graph of $\operatorname{vex}P$ is the symmetric, with respect to the main diagonal, of the Newton polygon, it follows from Proposition 2.10 that the Y_i and y_i in Theorem 3.1 coincide with the ones that are defined classically.

Theorem 3.1 is a "precise large deviation" version of the Newton-Puiseux theorem: we assume only the existence of asymptotic equivalents for the coefficients of $\mathcal{P}(\epsilon,\cdot)$, and derive the existence of asymptotic equivalents for the branches of

$\mathcal{P}(\epsilon,\cdot)$. The Newton-Puiseux algorithm is sometimes presented for asymptotic expansions, as in [Die68]. However, the equivalence between the two assertions of Theorem 3.1 does not seem to be classical. In particular, the asymptotics of some coefficients may be only known as being negligible: we require that $p_i \neq 0$ only for those i such that (i, P_i) is an exposed point of the epigraph of P.

Proof. We first prove $1 \Longrightarrow 2$. Let $Q = (\mathsf{Y} \oplus Y_1)\cdots(\mathsf{Y}\oplus Y_n)$. Then, $Q = \overline{Q}$, $\mathsf{R}(Q) = (Y_1 \leq \cdots \leq Y_n)$ and $Q_{n-i} = Y_1 \cdots Y_i$ for all $i = 1,\dots,n$. Since $\mathcal{Y}_1(\epsilon),\dots,\mathcal{Y}_n(\epsilon)$ are the roots of $\mathcal{P}(\epsilon, y) = 0$ counted with multiplicities, and $\mathcal{P}_n = 1$, it follows that $\mathcal{P}(\epsilon,\mathsf{Y}) = \prod_{i=1}^n (\mathsf{Y} - \mathcal{Y}_i(\epsilon))$. Hence, $(-1)^i\mathcal{P}_{n-i}$ is the sum of all products $\mathcal{Y}_{j_1}\cdots\mathcal{Y}_{j_i}$, where $j_1,\dots,j_i$ are pairwise distinct elements of $\{1,\dots,n\}$. By the properties of "$\simeq$" (stability by addition and multiplication), and since $\bigoplus_{j_1,\dots,j_i} Y_{j_1}\cdots Y_{j_i} = Y_1\cdots Y_i = Q_{n-i}$, we obtain that there exist $p_0,\dots,p_{n-1} \in \mathbb{C}$ such that $\mathcal{P}_j \simeq p_j \epsilon^{Q_j}$ for all $j = 0,\dots,n-1$. Putting $p_n = 1$, we also get $\mathcal{P}_n = 1 \simeq p_n\epsilon^{Q_n}$ since $Q_n = \mathbb{1}$. When $i = 1,\dots,n-1$ is such that $Y_i < Y_{i+1}$, $\mathcal{Y}_1\cdots\mathcal{Y}_i$ is the only leading term in the sum of all $\mathcal{Y}_{j_1}\cdots\mathcal{Y}_{j_i}$, and then $p_{n-i} = (-1)^i y_1\cdots y_i \neq 0$. Moreover, for $i = n$, either $Y_n \neq \mathbb{0}$, which implies that $p_0 = (-1)^n y_1 \cdots y_n \neq 0$, or $Y_n = \mathbb{0}$, which implies that $\mathcal{Y}_n = 0$, $\mathcal{P}_0 = 0$ and $Q_0 = \mathbb{0}$. This shows that $(c_1,\dots,c_n) = (Y_1,\dots,Y_n)$ and $P = Q$ are as in Point 2.

The remaining part of the theorem is obtained by a simple adaptation of the proof of the classical Newton-Puiseux theorem. When the $\mathcal{P}_j$ are only assumed to be continuous functions satisfying Point 2 of the theorem, it follows from (14,15,19), that $\mathsf{e}(\mathcal{P}) \geq P$, and since $P \geq \overline{P} = (\mathsf{Y}\oplus c_1)\cdots(\mathsf{Y}\oplus c_n)$, we get that $\mathsf{e}(\mathcal{P}) \geq (\mathsf{Y}\oplus c_1)\cdots(\mathsf{Y}\oplus c_n)$. In addition, from (17) and Point 2 of the theorem, we get that $\mathsf{e}(\mathcal{P})_{n-i} = P_{n-i} = \overline{P}_n c_1 \cdot\cdots\cdot c_i$ for all $i \in \{0,n\}\cup\{i\in\{1,\dots,n-1\} \mid c_i < c_{i+1}\}$, hence Lemma 2.11 yields $\overline{\mathsf{e}(\mathcal{P})} = \overline{P}$, therefore, $\mathsf{R}(\mathsf{e}(\mathcal{P})) = \mathsf{R}(P) = (c_1 \leq \cdots \leq c_n)$. Moreover, the first step of the Puiseux algorithm shows that, for all roots $c \neq \mathbb{0}$ of P with multiplicity k, there are exactly k continuous branches with leading exponent c. Indeed, when $c = c_i = \cdots = c_{i+k-1} \neq \mathbb{0}$, the change of variable $y = z\epsilon^c$, and the division of $\mathcal{P}$ by $\epsilon^{\widehat{P}(c)}$, transforms the equation $\mathcal{P}(\epsilon, y) = 0$ into an equation $\mathcal{Q}(\epsilon, z) = 0$, where $\mathcal{Q}(\cdot, z)$ extends continuously to 0 with $\mathcal{Q}(0,z) = p^{(i)}(z)$. Since $\widehat{P}(c) = P_j c^j$ implies that $n-i-k+1 \leq j \leq n-i+1$, and since either $i-1 = 0$ or $c_{i-1} < c_i$, we get that $p_{n-i+1} \neq 0$, hence $\deg p^{(i)} = n-i+1$. Similarly, we have either $i+k-1 = n$ or $c_{i+k-1} < c_{i+k}$. In the second case, we get $p_{n-i-k+1} \neq 0$, thus $\operatorname{val} p^{(i)} = n-i-k+1$. In the first case, $i+k-1 = n$, $c = c_n$, and $p_0 \neq 0$ or $P_0 = \mathbb{0}$. Since $P_0 = \mathbb{0}$ implies $c = c_n = \mathbb{0}$, which contradicts our assumption, we must have $p_0 \neq 0$, hence again $\operatorname{val} p^{(i)} = n-i-k+1$. Hence, $\deg p^{(i)} - \operatorname{val} p^{(i)} = k$ and the conclusion is obtained by the standard Lemma 3.2 below. Finally, if $c = \mathbb{0}$ is a root with multiplicity k, then $\operatorname{val} P = k$, $c_{n-k} < c_{n-k+1} = \mathbb{0}$, and $p_k \neq 0$. This implies that (for all $\epsilon > 0$ in a neighbourhood of 0) $\mathcal{P}(\epsilon,\cdot)$ is a polynomial with valuation k, hence it has 0 as a root with multiplicity k. □

Lemma 3.2. *Let $\mathcal{Q}(\epsilon,\mathsf{Y}) = \sum_{i=0}^n \mathcal{Q}_j(\epsilon)\mathsf{Y}^j$, where the $\mathcal{Q}_j$ are continuous functions of $\epsilon \in [0,\epsilon_0)$ and let $m = \deg\mathcal{Q}(0,\cdot)$. Then, for any open ball B containing the roots of $\mathcal{Q}(0,\cdot)$, there are m continuous branches $\mathcal{Z}_1,\dots,\mathcal{Z}_m$ defined in some interval $[0,\epsilon_1)$, with $0 < \epsilon_1 \leq \epsilon_0$, such that $\mathcal{Z}_1(\epsilon),\dots,\mathcal{Z}_m(\epsilon)$ are exactly the roots of $\mathcal{Q}(\epsilon,\cdot)$ in B counted with multiplicities. Moreover, the roots of $\mathcal{Q}(\epsilon,\cdot)$ that are outside B tend to infinity when ϵ goes to 0.*

Proof. We only sketch the proof, which is classical. By the Cauchy index theorem, if γ is any circle in $\mathbb{C}$ containing no roots of $\mathcal{Q}(\epsilon,\cdot)$, the number of roots of $\mathcal{Q}(\epsilon,\cdot)$ inside γ is $(2\pi i)^{-1}\int_\gamma \partial_z \mathcal{Q}(\epsilon,z)(\mathcal{Q}(\epsilon,z))^{-1}\,\mathrm{d}z$. By continuity of $\epsilon \mapsto \mathcal{Q}(\epsilon,\cdot)$, the number of roots of $\mathcal{Q}(\epsilon',\cdot)$ inside γ (counted with multiplicities) is constant for ϵ' in some neighbourhood of ϵ. Taking B as in the lemma, $\gamma = \partial B$, and $\epsilon = 0$, we get exactly m roots of $\mathcal{Q}(\epsilon',\cdot)$ in B for ϵ' in some interval $[0,\epsilon_1)$. Consider now a ball $B_R \supset B$ of radius R. For ϵ' small enough, the number of roots of $\mathcal{Q}(\epsilon',\cdot)$ in either B_R or B is equal to m, hence any root of $\mathcal{Q}(\epsilon',\cdot)$ outside B must be outside B_R. This shows that the roots of $\mathcal{Q}(\epsilon',\cdot)$ that do not belong to B go to infinity, when $\epsilon' \to 0$. Finally, by taking small balls around each root of $\mathcal{Q}(\epsilon,\cdot)$, with $0 \le \epsilon < \epsilon_1$, we see that the map which sends ϵ to the unordered m-tuple of roots of $\mathcal{Q}(\epsilon,\cdot)$ that belong to B, is continuous on $[0,\epsilon_1)$. By a selection theorem for unordered m-tuples depending continuously on a real parameter (see for instance [Kat95, Ch. II, Section 5, 2]), we derive the existence of the m continuous branches $\mathcal{Z}_1,\dots,\mathcal{Z}_m$. □

Theorem 3.1 says that “the leading exponents of the roots are the min-plus roots”.

Example 3.3. Consider $\mathcal{P}(\epsilon,\mathsf{Y}) = \mathsf{Y}^3 + \epsilon^5\mathsf{Y}^2 - \epsilon^6\mathsf{Y} + \epsilon^{13}$. The min-plus polynomial $P = \mathsf{e}(\mathcal{P})$ is the one of Figure 1, hence its roots are $c_1 = c_2 = 3$ and $c_3 = 7$. We have $p^{(1)} = p^{(2)} = \mathsf{Y}^3 - \mathsf{Y}$ and $p^{(3)} = -\mathsf{Y}+1$. Hence, $\mathcal{P}$ has 3 continuous branches around 0 with first order asymptotics: $\mathcal{Y}_1 \sim \epsilon^3$, $\mathcal{Y}^2 \sim -\epsilon^3$ and $\mathcal{Y}_3 \sim \epsilon^7$. Theorem 3.1 states in particular that we need not know the asymptotic expansions of all the coefficients of $\mathcal{P}(\epsilon,\mathsf{Y})$: for instance, if $\mathcal{P}(\epsilon,\mathsf{Y}) = \mathsf{Y}^3 + o(\epsilon^3)\mathsf{Y}^2 - \epsilon^6\mathsf{Y} + \epsilon^{13}$, the polynomials P and $p^{(1)}, p^{(2)}, p^{(3)}$ are unchanged, so that we still have 3 continuous branches with the same asymptotics has above.

Remark 3.4. If $\mathcal{A} \in \mathcal{C}^{n\times n}$ satisfies (20), the characteristic polynomial of $\mathcal{A}_\epsilon$, $\mathcal{P}(\epsilon,\mathsf{Y}) = \det(\mathsf{Y}I - \mathcal{A}_\epsilon)$ is an element of $\mathcal{C}[\mathsf{Y}]$, since $\mathcal{C}$ is a ring. Applying Theorem 3.1 to $\mathcal{P}$, we can obtain, under some additional assumptions, first order asymptotics for the eigenvalues of $\mathcal{A}_\epsilon$. The difficulty is that the coefficients $\mathcal{P}_j$ of $\mathcal{P}$ need not have first order asymptotics (even if $a_{ij} \neq 0$ for all i,j) due to cancellations. Of course if the coefficients of $\mathcal{A}_\epsilon$ have Puiseux series expansions in ϵ, the $\mathcal{P}_j$ also have Puiseux series expansions in ϵ and a fortiori first order asymptotics. However, if we only assume that $\mathcal{A} \in \mathcal{C}^{n\times n}$ satisfies (20), we obtain that the $\mathcal{P}_j$ satisfy the conditions $\mathcal{P}_n = 1$ and $\mathcal{P}_j(\epsilon) \simeq p_j\epsilon^{P_j}$ for some exponents $P_j \in \mathbb{R}_{\min}$ computed using the exponents A_{ij} (see Section 3.3). Hence, if the eigenvalues of $\mathcal{A}_\epsilon$ have first order asymptotics, Theorem 3.1 gives the exponents of these asymptotics as a function of the P_j.

3.3. **Majorisation inequalities for roots of min-plus polynomials.** The permanent of a matrix with coefficients in an arbitrary semiring $(S,\oplus,\otimes)$ can be defined as usual:

$$\mathrm{per}(A) = \bigoplus_{\sigma\in\mathfrak{S}_n} \bigotimes_{i=1}^{n} A_{i\sigma(i)} \enspace ,$$

where $\mathfrak{S}_n$ is the set of permutations of $\{1,\dots,n\}$. In particular, for any matrix $A \in \mathbb{R}_{\min}^{n\times n}$,

$$\mathrm{per}(A) = \min_{\sigma\in\mathfrak{S}_n} \sum_{i=1}^{n} A_{i\sigma(i)} \enspace ,$$

and the *formal characteristic polynomial* of A is the polynomial

$$\operatorname{per}(\mathsf{Y}I \oplus A) = \bigoplus_{\sigma\in\mathfrak{S}_n} \bigotimes_{i=1}^{n} (\mathsf{Y}\delta_{i\sigma(i)} \oplus A_{i\sigma(i)}) \in \mathbb{R}_{\min}[\mathsf{Y}] \enspace ,$$

where I is the identity matrix, and $\delta_{ij} = \mathbb{1}$ if $i = j$ and $\delta_{ij} = \mathbb{0}$ otherwise. The associated min-plus polynomial function will be called the *characteristic polynomial function* of A.

We next assume that $\mathcal{A} \in \mathcal{C}^{n\times n}$ satisfies (20) and that the eigenvalues $\mathcal{L}_\epsilon^i$ ($i = 1,\dots,n$) of $\mathcal{A}_\epsilon$ have first order asymptotics, $\mathcal{L}_\epsilon^i \sim \lambda_i\epsilon^{\Lambda_i}$. We relate, in that case, the Λ_i with the roots of the characteristic polynomial of A.

We need first to recall the classical definition of weak majorisation (see [MO79] for more background).

Definition 3.5. Let $u, v \in \mathbb{R}_{\min}^n$. Let $u_{(1)} \leq \cdots \leq u_{(n)}$ (resp. $v_{(1)} \leq \cdots \leq v_{(n)}$) denote the components of u (resp. v) in increasing order. We say that u is *weakly (super) majorised* by v, and we write $u \prec^{\mathrm{w}} v$, if the following conditions hold:

$$u_{(1)} \cdots u_{(k)} \geq v_{(1)} \cdots v_{(k)} \quad \forall k = 1,\dots,n \enspace .$$

In fact, the weak majorisation relation is only defined in [MO79] for vectors of $\mathbb{R}^n$. Here, it is convenient to define this notion for vectors of $\mathbb{R}_{\min}^n$. We also used the min-plus notations for homogeneity with the rest of the paper. The following lemma states a useful monotonicity property of the map which associates to a formal min-plus polynomial P its sequence of roots, $\mathsf{R}(P)$.

Lemma 3.6. *Let $P, Q \in \mathbb{R}_{\min}[X]$ be two formal polynomial of degree n. Then,*

$$P \geq Q \text{ and } P_n = Q_n \implies \mathsf{R}(P) \prec^{\mathrm{w}} \mathsf{R}(Q) \enspace . \tag{22}$$

Proof. From $P \geq Q$, we deduce $\overline{P} \geq \overline{Q}$. Let $\mathsf{R}(P) = (c_1(P) \leq \cdots \leq c_n(P))$ and $\mathsf{R}(Q) = (c_1(Q) \leq \cdots \leq c_n(Q))$ denote the sequence of roots of P and Q, respectively. Using $\overline{P} \geq \overline{Q}$, $\overline{P}_n = P_n = Q_n = \overline{Q}_n$ and (6), we get $c_1(P)\cdots c_k(P) = \overline{P}_{n-k}(\overline{P}_n)^{-1} \geq \overline{Q}_{n-k}(\overline{Q}_n)^{-1} = c_1(Q)\cdots c_k(Q)$, for all $k = 1,\dots,n$, that is $\mathsf{R}(P) \prec^{\mathrm{w}} \mathsf{R}(Q)$. □

We shall also need the following notion of genericity. We will say that a property $\mathscr{P}(y)$ depending on the variable $y = (y_1,\dots,y_n) \in \mathbb{C}^n$ holds for *generic values* of y if the set of elements $y \in \mathbb{C}^n$ such that the property $\mathscr{P}(y)$ is false is a proper algebraic variety. This means that there exists $Q \in \mathbb{C}[\mathsf{Y}_1,\dots,\mathsf{Y}_n] \setminus \{0\}$ such that $\mathscr{P}(y)$ is false if $Q(y) = 0$. When the parameter y will be obvious, we shall simply say that $\mathscr{P}$ is *generic* or holds *generically*. It is clear that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are both generic, then "$\mathscr{P}_1$ and $\mathscr{P}_2$" is also generic.

Since any polynomial $q = \sum_{i_1,\dots,i_n\in\mathbb{N}} q_{i_1,\dots,i_n}\mathsf{Y}_1^{i_1}\cdots\mathsf{Y}_n^{i_n} \in \mathbb{C}[\mathsf{Y}_1,\dots,\mathsf{Y}_n]$ in n indeterminates can be seen as an element of $\mathcal{C}[\mathsf{Y}_1,\dots,\mathsf{Y}_n]$ whose coefficients are constant with respect to ϵ, we have:

$$\mathsf{e}(q) = \bigoplus_{\substack{i_1,\dots,i_n\in\mathbb{N}\\ q_{i_1,\dots,i_n}\neq 0}} \mathsf{Y}_1^{i_1}\cdots\mathsf{Y}_n^{i_n} \in \mathbb{R}_{\min}[\mathsf{Y}_1,\dots,\mathsf{Y}_n] \enspace . \tag{23}$$

We also define, for all $Y \in \mathbb{R}_{\min}^n$:

$$q_Y^{\mathrm{Sat}}(\mathsf{Y}) := \sum_{\substack{i_1,\dots,i_n\in\mathbb{N}\\ \mathsf{e}(q)(Y_1,\dots,Y_n)=Y_1^{i_1}\cdots Y_n^{i_n}}} q_{i_1,\dots,i_n}\mathsf{Y}_1^{i_1}\cdots\mathsf{Y}_n^{i_n} \in \mathbb{C}[\mathsf{Y}_1,\dots,\mathsf{Y}_n] \enspace . \tag{24}$$

The following result is clear from the above definitions of $\mathsf{e}(q)$ and q_Y^{Sat}, since when $y \neq 0$, $\mathcal{Y} \simeq y\epsilon^Y \iff \mathcal{Y} \sim y\epsilon^Y$.

Lemma 3.7. *Let $q \in \mathbb{C}[\mathsf{Y}_1, \ldots, \mathsf{Y}_n]$ and let $Q = \mathsf{e}(q)$ and q_Y^{Sat} be defined by (23) and (24), respectively. Let $\mathcal{Y} \in \mathcal{C}^n$, $y \in \mathbb{C}^n$ and $Y \in \mathbb{R}^n_{\min}$ be such that $\mathcal{Y}_i \simeq y_i\epsilon^{Y_i}$ for $i = 1, \ldots, n$. Then,*

$$q(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \simeq q_Y^{\mathrm{Sat}}(y)\epsilon^{Q(Y)} \quad , \tag{25}$$

and for any fixed Y, we have an equivalence $\sim$ in (25) for generic values of $y \in \mathbb{C}^n$.

Theorem 3.8. *Let $\mathcal{A} \in \mathcal{C}^{n\times n}$ satisfy (20). Assume that the eigenvalues $\mathcal{L}^1_\epsilon, \ldots, \mathcal{L}^n_\epsilon$ of $\mathcal{A}_\epsilon$ (counted with multiplicities) have first order asymptotics, $\mathcal{L}^i_\epsilon \sim \lambda_i\epsilon^{\Lambda_i}$, and denote by $\Lambda = (\Lambda_1 \leq \cdots \leq \Lambda_n)$ the sequence of their exponents (counted with multiplicities). Let $\Gamma = (\gamma_1 \leq \cdots \leq \gamma_n)$ be the sequence of roots of the min-plus characteristic polynomial of A. Then,*

$$\Lambda \prec^{\mathrm{w}} \Gamma \quad , \tag{26}$$

and for generic values of $a = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\Lambda = \Gamma$.

Proof. Since $\mathcal{A} = \mathcal{A}_\epsilon \in \mathcal{C}^{n\times n}$, the characteristic polynomial of $\mathcal{A}$, $\mathcal{Q}(\epsilon, \mathsf{Y}) := \det(\mathsf{Y}I - \mathcal{A}_\epsilon)$ belongs to $\mathcal{C}[\mathsf{Y}]$. Let $Q = \mathsf{e}(\mathcal{Q}) \in \mathbb{R}_{\min}[\mathsf{Y}]$ and let $P = \mathrm{per}(\mathsf{Y}I \oplus A) \in \mathbb{R}_{\min}[\mathsf{Y}]$ be the min-plus characteristic polynomial of A. By (14,15), $\mathsf{e}(\mathcal{Q}) \geq \mathrm{per}(\mathsf{Y}I \oplus \mathsf{e}(-\mathcal{A}))$ and by (20) and (19), $\mathsf{e}(-\mathcal{A}) \geq A$. It follows that $Q = \mathsf{e}(\mathcal{Q}) \geq \mathrm{per}(\mathsf{Y}I \oplus A) = P$. Hence, from Lemma 3.6, we get that $\mathsf{R}(Q) \prec^{\mathrm{w}} \mathsf{R}(P) = \Gamma$. Moreover, by Theorem 3.1 applied to $\mathcal{Q}$, we get that $\mathsf{R}(Q) = \Lambda$, which finishes the proof of (26).

Let us show the genericity of the equality $\Lambda = \Gamma$. For all $a \in \mathbb{C}^{n\times n}$, we consider the k-th trace of a:

$$\mathrm{tr}_k(a) = \sum_{J\subset\{1,\ldots,n\},\, \#J=k} \left(\sum_{\sigma\in\mathfrak{S}_J} \mathrm{sgn}(\sigma) \prod_{j\in J} a_{j\sigma(j)} \right) \quad .$$

For all $A \in \mathbb{R}^{n\times n}_{\min}$, we also set

$$\mathrm{tr}_k(A) = \bigoplus_{J\subset\{1,\ldots,n\},\, \#J=k} \left(\bigoplus_{\sigma\in\mathfrak{S}_J} \bigotimes_{j\in J} A_{j\sigma(j)} \right) \quad . \tag{27}$$

Then, the coefficients of $\mathcal{Q}$ are given by $\mathcal{Q}_k(\epsilon) = (-1)^k \mathrm{tr}_{n-k}(\mathcal{A}_\epsilon)$, for $k = 0, \ldots, n-1$ and $\mathcal{Q}_n = 1$. The coefficients of P are given by $P_k = \mathrm{tr}_{n-k}(A)$, for $k = 0, \ldots, n-1$ and $P_n = \mathbb{1}$. By Lemma 3.7, we obtain that for any fixed (irreducible) matrix $A \in \mathbb{R}^{n\times n}_{\min}$, and any $\mathcal{A} \in \mathcal{C}^{n\times n}$ satisfying (20) with $a \in \mathbb{C}^{n\times n}$ and A, $\mathrm{tr}_k(\mathcal{A}_\epsilon) \sim (\mathrm{tr}_k)^{\mathrm{Sat}}_A(a)\epsilon^{\mathrm{tr}_k(A)}$ for generic values of $a \in \mathbb{C}^{n\times n}$. In particular, generically, $\mathcal{Q}_k(\epsilon)$ has first order asymptotics and $\mathsf{e}(\mathcal{Q}_k) = P_k$, for all $k = 0, \ldots, n$. This implies that $Q = P$, thus $\Lambda = \mathsf{R}(Q) = \mathsf{R}(P) = \Gamma$, generically. $\square$

Remark 3.9. Since a result of Burkard and Butkovič [BB03] shows that we can compute the min-plus characteristic polynomial function of a matrix in polynomial time (by solving $O(n)$ assignment problems), Theorem 3.8 shows that the sequence Λ of generic exponents of the eigenvalues can be computed in polynomial time. See also [Mur90, ABG04].

4. Critical values of min-plus matrices

4.1. **Schur complements and generalised circuit means.** We now construct another sequence $\beta = (\beta_1 \leq \cdots \leq \beta_n)$ using eigenvalues of min-plus matrices. First, we build by induction a finite sequence of min-plus square matrices A_ℓ and scalars $\alpha_\ell \in \mathbb{R}$, for $1 \leq \ell \leq k$, together with a partition $C_1 \cup \cdots \cup C_k = \{1, \dots, n\}$.

We start with $A_1 = A$. Then, for all $\ell \geq 1$, we define

$$\alpha_\ell = \rho_{\min}(A_\ell) \tag{28}$$

and we take for C_ℓ the set of critical nodes of A_ℓ. We build, as long as $C_1 \cup \cdots \cup C_\ell \neq \{1, \dots, n\}$, the min-plus Schur complement:

$$A_{\ell+1} = \mathrm{Schur}(C_\ell, \alpha_\ell, A_\ell) \enspace .$$

Due to the irreducibility of A, Lemma 2.14 shows that A_ℓ is irreducible, so that $C_\ell \neq \varnothing$. Hence, the algorithm stops at some index $k \leq n$. By Proposition 2.15, we get that $\alpha_1 < \cdots < \alpha_k$. We call $\alpha_1, \dots, \alpha_k$ the *critical values* of A. We define the *multiplicity* of the critical value α_ℓ as $\# C_\ell$. Repeating each critical value with its multiplicity, we obtain a sequence $\beta = (\beta_1 \leq \cdots \leq \beta_n)$ which will be called the *sequence of critical values counted with multiplicities*.

Let us give now a graph interpretation of the exponents α_ℓ. We set $C^0 = \varnothing$ and, for all $\ell = 1, \dots, k$,

$$C^\ell = C_1 \cup \ldots \cup C_\ell, \qquad N^\ell = \{1, \dots, n\} \setminus C^{\ell-1} \enspace .$$

For all paths p of $G(A)$ and all $\ell = 1, \dots, k$, we use the notations of Section 2.3 and:

$$|p|_A^\ell := |p|_A - \alpha_1 |p|_{C_1} - \cdots - \alpha_{\ell-1} |p|_{C_{\ell-1}} \enspace ,$$
$$|p|^\ell := |p| - |p|_{C_1} - \cdots - |p|_{C_{\ell-1}} = |p|_{N^\ell} \enspace .$$

Proposition 4.1. *The numbers α_ℓ defined in* (28) *satisfy:*

$$\alpha_\ell = \min \frac{|c|_A^\ell}{|c|^\ell} \enspace , \tag{29}$$

where the minimum is taken over all circuits c in $G(A)$ which are not included in $C^{\ell-1}$. Moreover, c is a critical circuit of A_ℓ if, and only if, there exists a circuit c' of $G(A)$ such that $c' \cap N^\ell = c$ and c' minimises (29).

Proof. Using repetitively Lemma 2.14, we get that for all circuits c of $G(A_\ell)$, $|c|_{A_\ell} = \min |c'|_A^\ell$, where the minimum is taken over all circuits c' of $G(A)$ such that $c' \cap N^\ell = c$. By the same arguments as in the proof of Proposition 2.15, we deduce the assertions of Proposition 4.1. □

Note that, as for Proposition 2.15, if c is included in $C^{\ell-1}$, that is if the denominator in (29) is zero, the numerator is necessarily nonnegative (by definition of $\alpha_{\ell-1}$).

We say that a circuit c of $G(A)$ is a *critical circuit of order* ℓ if $|c|_A^\ell = \alpha_\ell |c|^\ell$. We call *critical graph of order* ℓ the graph $G_\ell^c(A)$ whose nodes and arcs belong to critical circuits of order ℓ. Of course, $G^c(A) = G_1^c(A)$.

Proposition 4.2. *We have*

$$G_\ell^c(A) \subset G_{\ell+1}^c(A) \quad \ell = 1, \dots, k-1 \enspace . \tag{30}$$

which means that the nodes and arcs of $G_\ell^c(A)$ belong to $G_{\ell+1}^c(A)$.

Proof. If c is a critical circuit of order ℓ, then by definition $|c|_A^\ell = \alpha_\ell |c|^\ell$. If in addition $|c|^\ell = 0$, then $c \cap N^\ell = \varnothing$, whence $|c|_{C_\ell} = 0$ and $|c|^{\ell+1} = 0$. It follows that $|c|_A^{\ell+1} = |c|_A^\ell - \alpha_\ell |c|_{C_\ell} = 0 = \alpha_{\ell+1}|c|^{\ell+1}$, thus c is a critical circuit of order $\ell+1$. Otherwise, if $|c|^\ell \neq 0$, then c minimises (29) and since, by the arguments of the proof of Proposition 2.15, $|c \cap N^\ell|_{A_\ell} \leq |c|_A^\ell$, we obtain that $c' = c \cap N^\ell$ is a critical circuit of A_ℓ. By definition of C_ℓ, we get that the nodes of c' belong to C_ℓ, thus the nodes of c belong to C^ℓ, which shows $|c|^\ell = |c|_{C_\ell}$ or $|c|^{\ell+1} = 0$. Since $|c|_A^\ell = \alpha_\ell |c|^\ell$, we get $|c|_A^{\ell+1} = 0 = |c|^{\ell+1}$, and c is a critical circuit of order $\ell+1$. □

Let $\ell \in \{1, \dots, k\}$ and let D_ℓ denote the min-plus diagonal matrix such that $(D_\ell)_{jj} = \alpha_m$ if $j \in C_m$ with $m < \ell$, and $(D_\ell)_{jj} = \alpha_\ell$ if $j \in N^\ell$. For instance, if $n = 3$, $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $\alpha_1 = 2$ and $\alpha_2 = 4$, then $D_1 = \operatorname{diag}(2, 2, 2)$ and $D_2 = \operatorname{diag}(2, 4, 4)$. We set

$$\hat{A}_\ell = D_\ell^{-1} A \enspace .$$

We also set

$$G^c_\infty(A) = G^c_k(A), \quad \hat{A} = \hat{A}_k, \quad \text{and } D = D_k \enspace .$$

Lemma 4.3. *We have $A_\ell = \alpha_\ell \operatorname{Schur}(C^{\ell-1}, \hat{A}_\ell)$, for $\ell = 1, \dots, k$.*

Proof. We prove the lemma by induction on $\ell = 1, \dots, k$. Since $\hat{A}_1 = \alpha_1^{-1} A$ and $A_1 = A$, we get $A_1 = \alpha_1 \hat{A}_1$. If $A_\ell = \alpha_\ell \operatorname{Schur}(C^{\ell-1}, \hat{A}_\ell)$, then using (10) and (11), we get

$$\begin{aligned} A_{\ell+1} &= \operatorname{Schur}(C_\ell, \alpha_\ell, A_\ell) = \alpha_\ell \operatorname{Schur}(C_\ell, \alpha_\ell^{-1} A_\ell) \\ &= \alpha_\ell \operatorname{Schur}(C_\ell, \operatorname{Schur}(C^{\ell-1}, \hat{A}_\ell)) = \alpha_\ell \operatorname{Schur}(C^\ell, \hat{A}_\ell) \\ &= \alpha_\ell \operatorname{Schur}(C^\ell, D_\ell^{-1} D_{\ell+1} \hat{A}_{\ell+1}) \enspace . \end{aligned}$$

Since $(D_\ell^{-1} D_{\ell+1})_{jj} = \mathbb{1}$ for $j \in C^\ell$, and $(D_\ell^{-1} D_{\ell+1})_{jj} = \alpha_\ell^{-1}\alpha_{\ell+1}$ otherwise, it follows from (9) that $A_{\ell+1} = \alpha_{\ell+1} \operatorname{Schur}(C^\ell, \hat{A}_{\ell+1})$. □

Proposition 4.4. *For all $1 \leq \ell \leq k$, we have $G^c_\ell(A) = G^c(\hat{A}_\ell)$, $\hat{A}_\ell$ has min-plus eigenvalue $\mathbb{1}$, and the set of critical nodes of $\hat{A}_\ell$ is C^ℓ. Moreover, $G^c_\ell(A)$ and $G^c(\hat{A}) \cap C^\ell \times C^\ell$ (that is the restriction of $G^c(\hat{A})$ to the nodes of C^ℓ) have the same strongly connected components. In particular, $G^c_\infty(A) = G^c(\hat{A})$ and all the nodes of $\{1, \dots, n\}$ are critical for $\hat{A}$.*

Proof. For all circuits c and for all $\ell = 1, \dots, k$, we get by Proposition 4.1, $|c|_A^\ell \geq \alpha_\ell |c|^\ell$. Since, for all circuits $|c|_{\hat{A}_\ell} = |c|_A - \alpha_1 |c|_{C_1} - \cdots - \alpha_{\ell-1}|c|_{C_{\ell-1}} - \alpha_\ell |c|_{N^\ell} = |c|_A^\ell - \alpha_\ell |c|^\ell$, we get that $\rho(\hat{A}_\ell) \geq 0$. Moreover, c is a critical circuit of order ℓ if, and only if, $|c|_A^\ell = \alpha_\ell |c|^\ell$, which is equivalent to $|c|_{\hat{A}_\ell} = 0$. This shows that $\rho(\hat{A}_\ell) = 0$ and that c is a critical circuit of order ℓ if, and only if, c is a critical circuit of $\hat{A}_\ell$. It follows that $G^c_\ell(A) = G^c(\hat{A}_\ell)$. Since by Proposition 4.1, any critical circuit of A_ℓ is of the form $c' \cap N^\ell$ where c' is a critical circuit of order ℓ, the set C_ℓ of nodes of $G^c(A_\ell)$ is included in the set of nodes of $G^c_\ell(A)$. Using (30), we get by induction that C^ℓ is included in the set of nodes of $G^c_\ell(A)$. Conversely, since any critical circuit c' of order ℓ is such that $c' \cap N^\ell$ is a critical circuit of A_ℓ, and since the set of critical nodes of A_ℓ is C_ℓ, the set of nodes of $G^c_\ell(A)$ is included in $(\{1, \dots, n\} \setminus N^\ell) \cup C_\ell = C^\ell$, hence is equal to C^ℓ. Finally it is clear that, by definition of $\hat{A}_\ell$, $G^c(\hat{A}_\ell) \subset G^c(\hat{A})$, and since its set of nodes is C^ℓ, we get

$G^c(\hat{A}_\ell) \subset G^c(\hat{A}) \cap C^\ell \times C^\ell$. Conversely, since the restrictions of $\hat{A}$ and $\hat{A}_\ell$ to $C^\ell \times C^\ell$ are equal and since $\rho(\hat{A}_\ell) = \rho(\hat{A}) = 0$, any critical circuit of $\hat{A}$ with nodes in C^ℓ is critical for $\hat{A}_\ell$. It follows that the strongly connected components of $G^c(\hat{A}_\ell)$ and $G^c(\hat{A})$ are equal. □

Example 4.5. To illustrate the computation of the critical values, consider

$$A = \begin{bmatrix} \infty & 0 & \infty & \infty \\ 0 & \infty & 1 & \infty \\ 1 & \infty & \infty & 2 \\ \infty & \infty & 4 & 5 \end{bmatrix} . \tag{31}$$

We have $\alpha_1 = 0$, and the critical graph of A is composed of the circuit $(1 \to 2 \to 1)$. Thus, $C_1 = \{1, 2\}$. We have

$$\begin{aligned} A_2 &= \mathrm{Schur}(C_1, \alpha_1, A) \\ &= \begin{bmatrix} \infty & 2 \\ 4 & 5 \end{bmatrix} \oplus \begin{bmatrix} 1 & \infty \\ \infty & \infty \end{bmatrix} \begin{bmatrix} \infty & 0 \\ 0 & \infty \end{bmatrix}^* \begin{bmatrix} \infty & \infty \\ 1 & \infty \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix} . \end{aligned}$$

Hence, $\alpha_2 = \rho_{\min}(A_2) = 2$, with a unique associated critical circuit $(3 \to 3)$, and $C_2 = \{3\}$. (Recall our convention that Schur complements inherit their indices from the matrices from which they are defined, so that $(A_2)_{33} = 2$ is the top left entry of A_2.) We have $A_3 = \mathrm{Schur}(C_2, \alpha_2, A_2) = 5 \oplus 0^* 4 = 4$, hence, $\alpha_3 = 4$, with a unique associated critical circuit, $4 \to 4$, and $C_3 = \{4\}$.

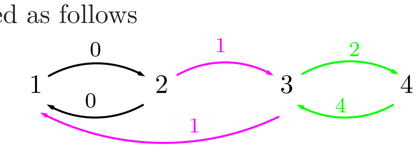
To determine the critical graphs $G^c_i(A)$, we use Proposition 4.4, which shows that $G^c_\ell(A) = G^c(\hat{A}_\ell)$. We already computed $G^c_1(A) = G^c(A)$. Since $D_2 = \mathrm{diag}(0, 0, 2, 2)$, and

$$\hat{A}_2 = D_2^{-1} A = \begin{bmatrix} \infty & 0 & \infty & \infty \\ 0 & \infty & 1 & \infty \\ -1 & \infty & \infty & 0 \\ \infty & \infty & 2 & 3 \end{bmatrix}$$

we deduce that $G^c_2(A) \setminus G^c_1(A)$ consists of the circuit $(1 \to 2 \to 3 \to 1)$. Finally, $D_4 = \mathrm{diag}(0, 0, 2, 4)$ and

$$\hat{A}_3 = D_3^{-1} A = \begin{bmatrix} \infty & 0 & \infty & \infty \\ 0 & \infty & 1 & \infty \\ -1 & \infty & \infty & 0 \\ \infty & \infty & 0 & 1 \end{bmatrix} \tag{32}$$

which shows that $G^c_3(A) \setminus G^c_2(A)$ consists of the circuit $(3 \to 4 \to 3)$. The critical graphs are represented as follows



Here, the graphs $G^c_\ell(A)$, for $\ell = 1, 2, 3$ are represented in black, magenta (medium gray), and green (light grey), respectively; for readability, a node or arc is drawn with the colour of the minimal graph $G^c_\ell(A)$ to which it belongs.

For such a small example, the critical circuits could be obtained by mere inspection. In general, $G^c_\ell(A) = G^c(\hat{A}_\ell)$ can be computed in polynomial time thanks to

Proposition 2.4, which shows that $G^c(\hat{A}_\ell)$ coincides with the union of the strongly connected components of $\mathrm{Sat}(\hat{A}_\ell, V)$, for any eigenvector V of $\hat{A}_\ell$.

4.2. **Majorisation inequalities for critical values.** We now state a second majorisation result, which should be compared with Theorem 3.8.

Theorem 4.6. *Consider an irreducible matrix $A \in \mathbb{R}_{\min}^{n\times n}$. Let $\Gamma = (\gamma_1 \leq \cdots \leq \gamma_n)$ be the sequence of roots of the min-plus characteristic polynomial of A and let $\beta = (\beta_1 \leq \cdots \leq \beta_n)$ be the sequence of critical values of A, repeated with multiplicities. Then,*

$$\Gamma \prec^{\mathrm{w}} \beta \ . \tag{33}$$

Proof. Let $P = \mathrm{per}(\mathsf{Y}I \oplus A)$ be the min-plus characteristic polynomial of A, $\Gamma = \mathsf{R}(P)$ and $Q = (\mathsf{Y} \oplus \beta_1)\cdots(\mathsf{Y} \oplus \beta_n)$. Let V be an eigenvector of $\hat{A}$ (for instance any column $\hat{A}^*_{\cdot j}$, since by Proposition 4.4, $\rho(\hat{A}) = \mathbb{1}$ and all the nodes of $\{1,\ldots,n\}$ are critical). Let $W = \mathrm{diag}\, V$. Since $\hat{A}V = V$, we get $W^{-1}\hat{A}W\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is the vector with all entries equal to $\mathbb{1}$. Therefore, $W^{-1}AW\mathbb{1} = D\mathbb{1}$, thus $(W^{-1}AW)_{ij} \geq \beta_i$ for all $i, j = 1,\ldots,n$. Using (27), we get

$$\mathrm{tr}_k(A) = \mathrm{tr}_k(W^{-1}AW) \geq \beta_1\cdots\beta_k \ . \tag{34}$$

Then,

$$\begin{aligned} P = \mathsf{Y}^n \oplus \mathrm{tr}_1(A)\mathsf{Y}^{n-1} \oplus \cdots \oplus \mathrm{tr}_n(A) &\geq \mathsf{Y}^n \oplus \beta_1\mathsf{Y}^{n-1} \oplus \cdots \oplus \beta_1\cdots\beta_n\mathsf{Y}^0 \\ &= (\mathsf{Y} \oplus \beta_1)\cdots(\mathsf{Y} \oplus \beta_n) = Q \ . \end{aligned}$$

From Lemma 3.6, we deduce $\mathsf{R}(P) \prec^{\mathrm{w}} \mathsf{R}(Q)$ and since $\Gamma = \mathsf{R}(P)$ and $\mathsf{R}(Q) = \beta$, we obtain (33). □

We next characterise the cases where the equality holds in (33). We say that a graph G has a *disjoint circuit cover* if there is a disjoint union of circuits containing all the nodes of G. This property, which is equivalent to the adjacency matrix of G having full *term rank* [BR91, Section 1.2], can be easily checked: it reduces to find a perfect matching (or to compute a matching of maximal cardinality) in a bipartite graph.

Theorem 4.7. *Consider an irreducible matrix $A \in \mathbb{R}_{\min}^{n\times n}$. Let $\Gamma = (\gamma_1 \leq \cdots \leq \gamma_n)$ be the sequence of roots of the min-plus characteristic polynomial of A, and let $\beta = (\beta_1 \leq \cdots \leq \beta_n)$ be the sequence of critical values of A repeated with multiplicities. For all $\ell \in \{1,\ldots,k\}$, where k is the number of critical values of A, the following assertions are equivalent:*

1. *$\gamma_j = \beta_j$ for $j \in \{\#C^{\ell-1}+1,\ldots,\#C^\ell\}$, and $\gamma_1\cdots\gamma_{\#C^{\ell-1}} = \beta_1\cdots\beta_{\#C^{\ell-1}}$;*
2. *$G^c_{\ell-1}(A)$ and $G^c_\ell(A)$ have a disjoint circuit cover.*

In Theorem 4.7, we use the convention that $G^c_0(A)$ is the empty graph and that it has a disjoint circuit cover. Recall also that $C^0 = \varnothing$.

The proof of Theorem 4.7 relies on the following lemma.

Lemma 4.8. *The equality*

$$\mathrm{tr}_{\#C^\ell}(A) = \beta_1\cdots\beta_{\#C^\ell} \tag{35}$$

holds if, and only if, $G^c_\ell(A)$ has a disjoint circuit cover.

Proof. Let us first assume that $G^c_\ell(A)$ has a disjoint circuit cover. Since by Proposition 4.4, the set of nodes of $G^c_\ell(A)$ is C^ℓ, there exists disjoint elementary circuits $c_1,\dots,c_q$ in $G^c_\ell(A)$ which cover all the nodes of C^ℓ. Let σ be the permutation of the nodes of C^ℓ which consists of the circuits $c_1,\dots,c_q$. We obtain, using (27):

$$\mathrm{tr}_{\#C^\ell}(A) \leq \bigotimes_{j\in C^\ell} A_{j\sigma(j)} = \beta_1\cdots\beta_{\#C^\ell} \bigotimes_{j\in C^\ell} (\hat{A}_\ell)_{j\sigma(j)} = \beta_1\cdots\beta_{\#C^\ell} \enspace ,$$

since, by Proposition 4.4, $c_1,\dots,c_q$ are critical circuits of $\hat{A}_\ell$ and $\rho(\hat{A}_\ell) = \mathbb{1}$. Since it follows from (34) that $\mathrm{tr}_{\#C^\ell}(A) \geq \beta_1\cdots\beta_{\#C^\ell}$, we have proved (35).

Conversely, let us assume that (35) holds. Let W be as in the proof of Theorem 4.6. By (27), there exists disjoint circuits $c_1,\dots,c_q$ of $G(A)$ such that $|c_1|+\cdots+|c_q| = \#C^\ell$ and $\mathrm{tr}_{\#C^\ell}(A) = \bigotimes_{j\in c_1\cup\cdots\cup c_q} A_{j\sigma(j)} = \bigotimes_{j\in c_1\cup\cdots\cup c_q} (W^{-1}AW)_{j\sigma(j)}$ where σ is the permutation of the nodes of C^ℓ consisting of the circuits $c_1,\dots,$ c_q. Since $W^{-1}AW\mathbb{1} \geq D\mathbb{1}$, we obtain that $\mathrm{tr}_{\#C^\ell}(A) \geq \bigotimes_{j\in c_1\cup\cdots\cup c_q} D_{jj}$. If $c_1\cup\cdots\cup c_q \neq C^\ell$, we obtain, using $\beta_n \geq \cdots \geq \beta_{\#C^\ell+1} > \beta_{\#C^\ell} \geq \cdots \geq \beta_1$, that $\mathrm{tr}_{\#C^\ell}(A) > \beta_1\cdots\beta_{\#C^\ell}$, a contradiction. Therefore, $c_1\cup\cdots\cup c_q = C^\ell$, and since $\beta_1\cdots\beta_{\#C^\ell} = \mathrm{tr}_{\#C^\ell}(A) = \bigotimes_{j\in C^\ell} A_{j\sigma(j)}$, we get $\bigotimes_{j\in c_1\cup\cdots\cup c_q} (\hat{A}_\ell)_{j\sigma(j)} = \mathbb{1}$. Since $\rho(\hat{A}_\ell) = \mathbb{1}$, the circuits $c_1,\dots,c_q$, which are critical for $\hat{A}_\ell$, are critical circuits of $G^c_\ell(A)$ (by Proposition 4.4). Hence, $G^c_\ell(A)$ has a disjoint circuit cover. ☐

Proof of Theorem 4.7. Let $P = \mathrm{per}(\mathsf{Y}I \oplus A)$ be the min-plus characteristic polynomial of A. By Lemma 2.11, we have

$$\overline{P}_{n-i} = \gamma_1\cdots\gamma_i \leq P_{n-i} = \mathrm{tr}_i(A) \enspace , \text{ with equality when } \gamma_i < \gamma_{i+1} \enspace . \tag{36}$$

We prove $2 \implies 1$. Assume that $G^c_{\ell-1}(A)$ and $G^c_\ell(A)$ have a disjoint circuit cover. Combining the inequality in (36) with (35), we get $\gamma_1\cdots\gamma_{\#C^\ell} \leq \beta_1\cdots\beta_{\#C^\ell}$. Similarly, $\gamma_1\cdots\gamma_{\#C^{\ell-1}} \leq \beta_1\cdots\beta_{\#C^{\ell-1}}$. Using (33), we get the reverse inequalities

$$\gamma_1\cdots\gamma_j \geq \beta_1\cdots\beta_j, \text{ for } j = 1,\dots,n \tag{37}$$

so that

$$\gamma_1\cdots\gamma_{\#C^\ell} = \beta_1\cdots\beta_{\#C^\ell} \enspace , \tag{38}$$

$$\gamma_1\cdots\gamma_{\#C^{\ell-1}} = \beta_1\cdots\beta_{\#C^{\ell-1}} \enspace . \tag{39}$$

Dividing (38) by (39), we get

$$\gamma_{\#C^{\ell-1}+1}\cdots\gamma_{\#C^\ell} = \beta_{\#C^{\ell-1}+1}\cdots\beta_{\#C^\ell} = \alpha_\ell^{\#C_\ell} \tag{40}$$

(recall that $\#C_\ell = \#C^\ell - \#C^{\ell-1}$). Taking $j = \#C^{\ell-1}+1$ in (37), and using (39), we get

$$\gamma_{\#C^{\ell-1}+1} \geq \beta_{\#C^{\ell-1}+1} = \alpha_\ell \enspace .$$

Since (γ_i) is nondecreasing, $\gamma_j \geq \gamma_{\#C^{\ell-1}+1} \geq \alpha_\ell$ holds for all $j \in \{\#C^{\ell-1}+1,\dots,\#C^\ell\}$, hence, if $\gamma_j > \alpha_\ell$ for some $j \in \{\#C^{\ell-1}+1,\dots,\#C^\ell\}$, we would have $\gamma_{\#C^{\ell-1}+1}\cdots\gamma_{\#C^\ell} > \alpha_\ell^{\#C_\ell}$, contradicting (40). Therefore, $\gamma_{\#C^{\ell-1}+1} = \cdots = \gamma_{\#C^\ell} = \alpha_\ell = \beta_{\#C^{\ell-1}+1} = \cdots = \beta_{\#C^\ell}$.

We next prove $1 \implies 2$. By assumption, (38) and (39) hold. Taking $j = \#C^\ell+1$ in (37) and using (38), we have $\gamma_{\#C^\ell+1} \geq \beta_{\#C^\ell+1}$. Since $\beta_{\#C^\ell+1} > \beta_{\#C^\ell} = \gamma_{\#C^\ell}$, we have $\gamma_{\#C^\ell+1} > \gamma_{\#C^\ell}$, so the equality case in (36) yields

$$\gamma_1\cdots\gamma_{\#C^\ell} = \mathrm{tr}_{\#C^\ell}(A) \enspace . \tag{41}$$

Taking now $j = \#C^{\ell-1} - 1$ in (37), and using (39), we get $\beta_{\#C^{\ell-1}} \geq \gamma_{\#C^{\ell-1}}$, hence, $\gamma_{\#C^{\ell-1}+1} = \beta_{\#C^{\ell-1}+1} > \beta_{\#C^{\ell-1}} \geq \gamma_{\#C^{\ell-1}}$, and the equality case in (36) yields

$$\gamma_1 \cdots \gamma_{\#C^{\ell-1}} = \mathrm{tr}_{\#C^{\ell-1}}(A) \ . \tag{42}$$

It follows from Lemma 4.8, and from (38), (39), (41) and (42), that $G^c_\ell(A)$ and $G^c_{\ell-1}(A)$ have disjoint circuits covers. □

Corollary 4.9. *If $G^c_{\ell-1}(A)$ and $G^c_\ell(A)$ have a disjoint circuit cover, then α_ℓ is a root of multiplicity $\#C_\ell$ of the min-plus characteristic polynomial of A.*

Proof. Since $\gamma_j = \beta_j = \alpha_\ell$ for $j \in \{\#C^{\ell-1}+1, \dots, \#C^\ell\}$, α_ℓ is a root of multiplicity at least $\#C^\ell - \#C^{\ell-1} = \#C_\ell$ of the characteristic polynomial of A. Moreover, we showed in the proof of "$1 \implies 2$" of Theorem 4.7 that $\gamma_{\#C^{\ell+1}} > \gamma_{\#C^\ell}$ and $\gamma_{\#C^{\ell-1}+1} > \gamma_{\#C^{\ell-1}}$. Thus, α_ℓ is a root of multiplicity exactly $\#C_\ell$ of the characteristic polynomial of A. □

5. Asymptotics of eigenvalues

5.1. **Statement and illustration of the result.** We next show that under some non-degeneracy conditions, the first order asymptotics of the eigenvalues of $\mathcal{A}_\epsilon$ are given by the critical values of A. If G is any graph with set of nodes $1, \dots, n$, and if $b \in \mathbb{C}^{n\times n}$, the matrix b^G is defined by

$$(b^G)_{ij} = \begin{cases} b_{ij} & \text{if } (i,j) \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be either the critical graph of $\hat{A}$ or the saturation graph $\mathrm{Sat}(\hat{A}, V)$, for any eigenvector V of $\hat{A}$ (since by Proposition 4.4, all the nodes $1, \dots, n$ belong to the critical graph of $\hat{A}$, we can take for V any column of $\hat{A}^*$).

We construct the following conventional Schur complements:

$$s^1 = a^G, \qquad s^\ell = \mathrm{Schur}(C^{\ell-1}, s^1),\ \ell = 2, \dots, k \ . \tag{43}$$

The Schur complement s^ℓ is well defined as soon as the matrix

$$r^\ell = a^G_{C^{\ell-1}, C^{\ell-1}} \tag{44}$$

is invertible (we adopt the convention that r^1 is the empty matrix, and is invertible). We shall also need the following matrix:

$$t^\ell = s^\ell_{C_\ell C_\ell} \ . \tag{45}$$

When both s^ℓ and $s^{\ell-1}$ are well defined, $t^{\ell-1}$ is invertible and we can compute s^ℓ from $s^{\ell-1}$ thanks to (11):

$$s^\ell = \mathrm{Schur}(C_{\ell-1}, s^{\ell-1}) \ .$$

We say that a function of ϵ, $f(\epsilon)$, is *of order* $\omega(\epsilon^\alpha)$ if $\lim_{\epsilon\to 0} |f(\epsilon)\epsilon^{-\alpha}| = +\infty$.

Theorem 5.1 (Generalised Lidskiĭ-Višik-Ljusternik theorem)**.** *Let s^ℓ, r^ℓ, t^ℓ, $\ell = 1, \dots, k$ be constructed as in (43,44,45) with $G = G^c(\hat{A})$ or equivalently with $G = \mathrm{Sat}(\hat{A}, V)$ for some eigenvector V of $\hat{A}$. Assume that the matrix r^ℓ is invertible for some $1 \leq \ell \leq k$, and let $\lambda^\ell_1, \dots, \lambda^\ell_{m_\ell}$ denote the non-zero eigenvalues of t^ℓ (here and in the sequel, eigenvalues are repeated with multiplicities). Then, the eigenvalues of $\mathcal{A}_\epsilon$ can be grouped in*

(1) *m_ℓ eigenvalues with asymptotic expansions*

$$\mathcal{L}_\epsilon^{\ell,j} \sim \lambda_j^\ell \epsilon^{\alpha_\ell}, \quad 1 \leq j \leq m_\ell \ , \tag{46}$$

(2) *$\#C^{\ell-1}$ eigenvalues of order $\omega(\epsilon^{\alpha_\ell})$,*
(3) *$\#N^\ell - m_\ell$ eigenvalues of order $o(\epsilon^{\alpha_\ell})$.*

In particular, when $t^1, \dots, t^k$ all are invertible, for all $1 \leq \ell \leq k$, $\mathcal{A}_\epsilon$ has exactly $\#C_\ell$ eigenvalues of order ϵ^{α_ℓ}, whose asymptotics are given by (46)*.*

We prove Theorem 5.1 in Section 5.2.

By Proposition 2.4, the saturation graph $\mathrm{Sat}(\hat{A}, V)$ (defined in Section 2.1) and the critical graph $G^c(\hat{A})$ have the same strongly connected components. This explains why, in Theorem 5.1, one can use either the graph $G = G^c(\hat{A})$ or the graph $G = \mathrm{Sat}(\hat{A}, V)$.

The following result, that we also prove in Section 5.2, shows that the assumptions of the theorem are generically satisfied, if we assume that the critical graphs have disjoint circuit covers:

Proposition 5.2. *Let $\ell = 1, \dots, k$. Assume that $G^c_{\ell-1}(A)$ and $G^c_\ell(A)$ have disjoint circuit covers. Then, r^ℓ and t^ℓ are generically invertible, so that the number of eigenvalues of $\mathcal{A}_\epsilon$ with an equivalent of the form $\lambda\epsilon^{\alpha_\ell}$, where $\lambda \in \mathbb{C} \setminus \{0\}$, is generically $\#C_\ell$.*

Example 5.3. To illustrate Theorem 5.1, consider the matrix

$$\mathcal{A}_\epsilon = \begin{bmatrix} \epsilon & 1 & \epsilon^4 \\ 0 & \epsilon & \epsilon^{-2} \\ \epsilon & \epsilon^2 & 0 \end{bmatrix} \ , \tag{47}$$

so that $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$, with

$$a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ , \quad A = \begin{bmatrix} 1 & 0 & 4 \\ \infty & 1 & -2 \\ 1 & 2 & \infty \end{bmatrix} \ .$$

We have $\rho_{\min}(A) = -1/3$, and $G^c(A)$ consists of the critical circuit:

$$\begin{tikzcd} 1 \arrow[r, bend left, "0"] & 2 \arrow[r, bend left, "-2"] & 3 \arrow[ll, bend left, "1"] \end{tikzcd}$$

so that the construction of the critical classes stops with $C_1 = \{1, 2, 3\}$ and $k = 1$. Then, $G^c(\hat{A}) = G^c_1(A) = G^c(A)$ covers all the nodes (see Proposition 4.4), hence,

$$s^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \ .$$

Since the spectrum of s^1 is $\{1, j, j^2\}$, Theorem 5.1 shows that the spectrum of $\mathcal{A}_\epsilon$ consists of the three eigenvalues

$$\mathcal{L}^1_\epsilon \sim \epsilon^{-1/3}, \mathcal{L}^2_\epsilon \sim j\epsilon^{-1/3}, \mathcal{L}^3_\epsilon \sim j^2\epsilon^{-1/3}.$$

Example 5.4. To give an example in which different exponents appear, consider

$$\mathcal{A}_\epsilon = \begin{bmatrix} \cdot & a_{12} & \cdot & \cdot \\ a_{21} & \cdot & \epsilon a_{23} & \cdot \\ \epsilon a_{31} & \cdot & \cdot & \epsilon^2 a_{34} \\ \cdot & \cdot & \epsilon^4 a_{43} & \epsilon^5 a_{44} \end{bmatrix} ,$$

where $a_{ij} \in \mathbb{C}$, and "$\cdot$" denotes a zero entry. The associated matrix of exponents A is given by (31), and we saw in Example 4.5 that the critical values of A are $\alpha_1 = 0$, $\alpha_2 = 2$, $\alpha_3 = 4$, with $C_1 = \{1, 2\}$, $C_2 = \{3\}$, $C_3 = \{4\}$. The critical graph $G = G^c(\hat{A})$ of the matrix $\hat{A} = \hat{A}_3$ of (32) was represented in Example 4.5. Thus,

$$s^1 = a^G = \begin{bmatrix} \cdot & a_{12} & \cdot & \cdot \\ a_{21} & \cdot & a_{23} & \cdot \\ a_{31} & \cdot & \cdot & a_{34} \\ \cdot & \cdot & a_{43} & \cdot \end{bmatrix} .$$

The eigenvalues of the matrix $t^1 = \left[\begin{smallmatrix} 0 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right]$ are the square roots of $a_{12}a_{21}$. Let us assume that $a_{12}a_{21} \neq 0$. Then, Theorem 5.1 shows that $\mathcal{A}_\epsilon$ has two eigenvalues with asymptotics of the form $\mathcal{L}_\epsilon \sim \xi$, where $\xi^2 = a_{12}a_{21}$. Moreover,

$$s^2 = \mathrm{Schur}(\{1, 2\}, s^1) = \begin{bmatrix} -a_{31}a_{21}^{-1}a_{23} & a_{34} \\ a_{43} & \cdot \end{bmatrix} , \quad t^2 = -a_{31}a_{21}^{-1}a_{23} .$$

If we assume additionally that $a_{31}a_{23} \neq 0$, Theorem 5.1 shows that $\mathcal{A}_\epsilon$ has an eigenvalue with asymptotics $\mathcal{L}_\epsilon \sim -a_{31}a_{21}^{-1}a_{23}\epsilon^2$. Finally, as soon as the matrix r^3 is invertible, i.e., as soon as $\det r^3 = a_{12}a_{23}a_{31} \neq 0$, the Schur complement

$$t^3 = s^3 = \mathrm{Schur}(\{1, 2, 3\}, s^1) = a_{43}a_{23}^{-1}a_{21}a_{31}^{-1}a_{34} .$$

is well defined. (When s^2 is well defined, that is $a_{12}a_{21} \neq 0$, and $a_{31}a_{23} \neq 0$, we may obtain equivalently t^3 as $\mathrm{Schur}(\{3\}, s^2)$.) Thus, when $a_{12}a_{23}a_{31} \neq 0$ and $a_{43}a_{21}a_{34} \neq 0$, Theorem 5.1 shows that $\mathcal{A}_\epsilon$ has an eigenvalue with asymptotics $\mathcal{L}_\epsilon \sim a_{43}a_{23}^{-1}a_{21}a_{31}^{-1}a_{34}\epsilon^4$.

5.2. **Proof of Theorem 5.1 and Proposition 5.2.** For the proof of Theorem 5.1, we need to use the following lemma, which follows readily from the definition of determinants.

Lemma 5.5. *If $b, \tilde{b} \in \mathbb{C}^{n\times n}$ have two digraphs $G(b)$ and $G(\tilde{b})$ whose circuits (or equivalently, whose strongly connected components) are the same, and if $b_{ij} = \tilde{b}_{ij}$ for all arcs (i, j) belonging to circuits of $G(b)$ or $G(\tilde{b})$, then, $\det b = \det \tilde{b}$.*

Let V be an eigenvector of $\hat{A}$ and let $\mathrm{Sat} = \mathrm{Sat}(\hat{A}, V)$. The change of variables $\lambda = \mu\epsilon^{\alpha_\ell}$, for some $1 \leq \ell \leq k$, transforms the characteristic polynomial of $\mathcal{A}_\epsilon$ into

$$\det(\mu\epsilon^{\alpha_\ell} I - \mathcal{A}_\epsilon) = \det(\epsilon^{D_\ell}) \det(\mu\epsilon^{\alpha_\ell}\epsilon^{D_\ell^{-1}} I - \epsilon^{D_\ell^{-1}}\mathcal{A}_\epsilon) = \det(\epsilon^{D_\ell})\mathcal{P}(\epsilon, \mu)$$
$$\text{where } \mathcal{P}(\epsilon, \mu) = \det(\mu\epsilon^{\alpha_\ell}\epsilon^{D_\ell^{-1}} I - \epsilon^{D_\ell^{-1}}\epsilon^{\mathrm{diag}(V)^{-1}}\mathcal{A}_\epsilon\epsilon^{\mathrm{diag}(V)}) .$$

If $C \subset L$ are finite sets, we denote by E_C^L the $L \times L$ diagonal matrix such that

$$(E_C^L)_{ii} = \begin{cases} 1 & \text{for } i \in C , \\ 0 & \text{for } i \in L \setminus C . \end{cases}$$

If $L = \{1, \dots, n\}$, we shall simply write E_C instead of E^L_C. We have

$$\epsilon^{D^{-1}}\epsilon^{\operatorname{diag}(V)^{-1}}\mathcal{A}_\epsilon\epsilon^{\operatorname{diag}(V)} \underset{\epsilon\to 0}{\longrightarrow} a^{\mathrm{Sat}}, \qquad \epsilon^{D_\ell^{-1}}\epsilon^{D} \underset{\epsilon\to 0}{\longrightarrow} E_{C^\ell} \ , \quad \text{and}$$

$$\epsilon^{\alpha_\ell}\epsilon^{D_\ell^{-1}} \underset{\epsilon\to 0}{\longrightarrow} E_{N^\ell} \ , \tag{48}$$

hence $\mathcal{P}(\epsilon,\mu) \longrightarrow_{\epsilon\to 0} \mathcal{P}(0,\mu)$, where

$$\mathcal{P}(0,\mu) = \det(\mu E_{N^\ell} - E_{C^\ell} a^{\mathrm{Sat}}) \ .$$

Since Sat and $G^c(\hat{A})$ have the same strongly connected components (by Proposition 2.4), Lemma 5.5 yields:

$$\mathcal{P}(0,\mu) = \det(\mu E_{N^\ell} - E_{C^\ell} a^{G^c(\hat{A})}) \ .$$

The same arguments also show that the invertibility of the matrix r^ℓ is independent of the choice of $G = \mathrm{Sat}$ or $G = G^c(\hat{A})$ in (43). Hence, if s^ℓ, r^ℓ, t^ℓ are constructed as in (43,44,45) with either $G = \mathrm{Sat}$ or $G = G^c(\hat{A})$, and if r^ℓ is invertible, then

$$\mathcal{P}(0,\mu) = \mu^{\#N^{\ell+1}} \det(\mu E^{C^\ell}_{C_\ell} - a^G_{C^\ell,C^\ell}) = \mu^{\#N^{\ell+1}} \det(-r^\ell)\det(\mu I - t^\ell) \ .$$

From Lemma 3.2 applied to $\mathcal{P}(\epsilon, \mathsf{Y})$, there exists $\#N^\ell$ continuous functions $\epsilon \mapsto \mathcal{L}^{m,j}_\epsilon$, with $j = 1, \dots, \#C_m$ and $m = \ell, \dots, k$, such that $\mathcal{L}^{m,j}_\epsilon$ are the roots of $\mathcal{P}(\epsilon, \mathsf{Y})$ for all ϵ small enough. Hence, $\mathcal{L}^{\ell,j}_0$ are the eigenvalues of t^ℓ and $\mathcal{L}^{m,j}_0 = 0$ for $m > \ell$. The other roots of $\mathcal{P}(\epsilon, \mathsf{Y})$ tend to infinity. This shows Theorem 5.1.

We finally prove Proposition 5.2. If the set of nodes of $G^c_{\ell-1}(A)$ can be covered by disjoint circuits, it follows from Proposition 4.4 that these circuits also belong to $G^c(\hat{A}) \cap C^{\ell-1} \times C^{\ell-1}$. By definition of r^ℓ, for generic values of $a = (a_{ij})$, these circuits belong to the graph of r^ℓ, which implies that the determinant of r^ℓ is generically non-zero. Thus, r^ℓ is generically invertible. The same argument shows that if $G^c_\ell(A)$ can be covered by disjoint circuits, $r^{\ell+1}$ is generically invertible, and since $t^\ell = \mathrm{Schur}(C^{\ell-1}, r^{\ell+1})$ is the Schur complement of the generically invertible $C^{\ell-1} \times C^{\ell-1}$ submatrix of $r^{\ell+1}$, namely r^ℓ, in the generically invertible matrix $r^{\ell+1}$, t^ℓ must also be generically invertible. Thus, $m_\ell = \#C_\ell$ generically in Theorem 5.1.

6. Asymptotics of eigenvectors

6.1. **Statement and illustration of the result.** We now consider eigenvectors.

Theorem 6.1. *Let s^ℓ, r^ℓ, t^ℓ, $\ell = 1, \dots, k$ be constructed as in Theorem 5.1. Assume that the matrix r^ℓ is invertible, for some $1 \le \ell \le k$, that $\mu \neq 0$ is a simple eigenvalue of t^ℓ, and let V be any eigenvector of $\hat{A}_\ell$. Then, the equation*

$$(\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell, V)})w = 0, \tag{49}$$

has a unique solution $w = (w_j) \in \mathbb{C}^n \backslash \{0\}$ up to a multiplicative constant. Moreover, there is a unique eigenvalue $\mathcal{L}_\epsilon$ with asymptotics $\mathcal{L}_\epsilon \sim \mu\epsilon^{\alpha_\ell}$, and if $w_i \neq 0$, any eigenvector $\mathcal{V}_\epsilon$ associated to this $\mathcal{L}_\epsilon$ satisfies $(\mathcal{V}_\epsilon)_i \neq 0$ for ϵ small enough, and

$$\frac{(\mathcal{V}_\epsilon)_j}{(\mathcal{V}_\epsilon)_i} \simeq \frac{w_j\epsilon^{V_j}}{w_i\epsilon^{V_i}} \ , \ \text{for } j \in \{1, \dots, n\} \ . \tag{50}$$

We prove Theorem 6.1 in Section 6.2.

Example 6.2. To illustrate Theorem 6.1, let us pursue the analysis of Example 5.3. We already showed that the eigenvalues of the matrix (47) have asymptotic equivalents of the form $\xi\epsilon^{-1/3}$, where ξ is a cubic root of 1. When $\mu = \xi$, any solution of (49) (with $\ell = 1$), is proportional to $w = [1, \xi, \xi^2]^T$. Since A has a unique critical class, $C_1 = \{1, 2, 3\}$, by Theorem 2.3, A has a unique eigenvector, up to a scalar factor, and we can take $V = [0, -1/3, 4/3]^T = \hat{A}^*_{\cdot,1}$. Theorem 6.1 shows that any eigenvector $\mathcal{V}_\epsilon$ associated to the eigenvalue $\xi\epsilon^{-1/3}$ is equivalent to

$$[1, \xi\epsilon^{-1/3}, \xi^2\epsilon^{4/3}]^T \quad ,$$

up to a scalar factor.

When $w_j = 0$, Theorem 6.1 gives a poor information on the asymptotics of $(\mathcal{V}_\epsilon)_j$. Moreover, when $\hat{A}_\ell$ has several critical classes (so that the eigenvector V is non unique) the non-zero character of w_j depends in a critical way of the eigenvector V which is selected.

Example 6.3. The following example illustrates the importance of the choice of the eigenvector V in Theorem 6.1. Consider

$$\mathcal{A}_\epsilon = \begin{bmatrix} 1 & \epsilon & \epsilon^3 \\ -2\epsilon & \epsilon^2 & \cdot \\ \epsilon^3 & \cdot & 2\epsilon^2 \end{bmatrix}$$

which is such that $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$ with

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & \infty \\ 3 & \infty & 2 \end{bmatrix}, \ a = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & \cdot \\ 1 & \cdot & 2 \end{bmatrix} \quad .$$

We have $\alpha_1 = \rho_{\min}(A) = 0$, with a unique critical circuit $(1 \to 1)$. Hence, $C_1 = \{1\}$, and

$$A_2 = \mathrm{Schur}(\{1\}, A) = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \quad .$$

Thus, $\alpha_2 = 2$, $C_2 = \{2, 3\}$. We have

$$\hat{A} = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & \infty \\ 1 & \infty & 0 \end{bmatrix} .$$

Since the critical graph of $\hat{A}$, which is the union of the complete graph on $\{1, 2\}$, and of the loop $(3 \to 3)$, has two strongly connected components, $\{1, 2\}$, and $\{3\}$, the eigenspace of $\hat{A}$ is spanned by the two vectors $\hat{A}^*_{\cdot,i}$, $i = 1, 3$. Let us take

$$V = \hat{A}^*_{\cdot,3} = [3, 2, 0]^T \quad ,$$

for which the saturation graph is obtained by adding the arc $(1 \to 3)$ to the critical graph of $\hat{A}$. Taking $G = \mathrm{Sat}(\hat{A}, V)$ in (43), we get

$$s^1 = a^{\mathrm{Sat}(\hat{A},V)} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix} \quad . \tag{51}$$

Since $t^1 = 1$, Theorem 5.1 shows that $\mathcal{A}_\epsilon$ has a root with asymptotics $\mathcal{L}_\epsilon \sim 1$, and since $t^2 = s^2 = \mathrm{Schur}(\{1\}, s^1) = \left[\begin{smallmatrix} 3 & 2 \\ 0 & 2 \end{smallmatrix}\right]$ has roots $2, 3$, $\mathcal{A}_\epsilon$ has two eigenvalues with respective asymptotics $\mathcal{L}_\epsilon \sim 2\epsilon^2$, and $\mathcal{L}_\epsilon \sim 3\epsilon^2$. Let us compute for instance the

asymptotics of the eigenvector $\mathcal{V}_\epsilon$ associated to $\mathcal{L}_\epsilon \sim 2\epsilon^2$, using Theorem 6.1 with $\ell = 2$ and $\mu = 2$ (thus $\hat{A}_2 = \hat{A}$). With the previous choice of V, we need to solve the system (49), which, by (51), specialises to

$$w_1 + w_2 + w_3 = 0,\ -2w_1 - w_2 = 0,\ 0 = 0 \ .$$

All the solutions of this system are proportional to $w = [1, -2, 1]^T$. Thus, Theorem 6.1 shows that up to a multiplicative constant,

$$\mathcal{V}_\epsilon \sim [\epsilon^3, -2\epsilon^2, 1]^T \ .$$

Consider now the alternative choice of V:

$$V = \hat{A}^*_{\cdot,1} = [0, -1, 1]^T \ .$$

Then, $\mathrm{Sat}(\hat{A}_2, V)$ is obtained by adding the arc $(3 \to 1)$ to the critical graph of $\hat{A}$. Theorem 6.1 yields that $(\mathcal{V}_\epsilon)_i \simeq w_i \epsilon^{V_i}$, where

$$w_1 + w_2 = 0,\ -2w_1 - w_2 = 0,\ w_1 = 0$$

and since all the solutions w are proportional to $[0, 0, 1]^T$, we learn only from (50) that $(\mathcal{V}_\epsilon)_1/(\mathcal{V}_\epsilon)_3 \simeq 0\epsilon^{-1}$, and $(\mathcal{V}_\epsilon)_2/(\mathcal{V}_\epsilon)_3 \simeq 0\epsilon^{-2}$, a very poor information.

Remark 6.4. When μ is not a simple root of t^ℓ, the first order asymptotics of the eigenvector may be ruled by higher order terms in the expansions of the entries of $\mathcal{A}_\epsilon$, see [ABG98] for a special case.

6.2. **Proof of Theorem 6.1.** We first observe that by Theorem 5.1, there is only one eigenvalue $\mathcal{L}_\epsilon$ of $\mathcal{A}_\epsilon$ equivalent to $\mu\epsilon^{\alpha_\ell}$. Then the associated eigenvector, $\mathcal{V}_\epsilon$, is unique, up to a multiplicative constant, since for ϵ small enough, $\mathcal{L}_\epsilon$ is a simple eigenvalue of $\mathcal{A}_\epsilon$.

To prove Theorem 6.1, we perform the change of variables $\mathcal{V}_\epsilon = \epsilon^{\mathrm{diag}\, V}\mathcal{W}_\epsilon$ and $\mathcal{L}_\epsilon = \mathcal{M}_\epsilon \epsilon^{\alpha_\ell}$, where $\mathcal{M}_\epsilon \to \mu$ when $\epsilon \to 0$. After multiplying $\mathcal{V}_\epsilon$ by a constant, we may assume that $\sum_{1\leq j\leq n} |(\mathcal{W}_\epsilon)_j| = 1$. From $\mathcal{A}_\epsilon\mathcal{V}_\epsilon = \mathcal{L}_\epsilon\mathcal{V}_\epsilon$, we get

$$\epsilon^{D_\ell^{-1}}\epsilon^{(\mathrm{diag}\, V)^{-1}}\mathcal{A}_\epsilon\epsilon^{\mathrm{diag}\, V}\mathcal{W}_\epsilon = \mathcal{M}_\epsilon\epsilon^{D_\ell^{-1}}\epsilon^{\alpha_\ell}\mathcal{W}_\epsilon \ ,$$

where $\epsilon^{D_\ell^{-1}}\epsilon^{\mathrm{diag}(V)^{-1}}\mathcal{A}_\epsilon\epsilon^{\mathrm{diag}\, V} \to a^{\mathrm{Sat}(\hat{A}_\ell, V)}$ when $\epsilon \to 0$. Together with (48), this implies that any limit point w of $\mathcal{W}_\epsilon$ when $\epsilon \to 0$ satisfies

$$a^{\mathrm{Sat}(\hat{A}_\ell, V)} w = \mu E_{N^\ell} w, \quad \text{and } |w_1| + \cdots + |w_n| = 1 \ . \tag{52}$$

To show that the solution w of (52) is unique, up to the multiplication by a complex number of modulus 1, we shall prove that $\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell, V)}$ has rank $n-1$.

Since, by Proposition 2.4, $\mathrm{Sat}(\hat{A}_\ell, V)$ and $G^c(\hat{A}_\ell) = G^c_\ell(A)$ have the same strongly connected components, applying Lemma 5.5 to the matrices $b = b(\lambda) = \lambda E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell, V)}$ and $\tilde{b} = \tilde{b}(\lambda) = \lambda E_{N^\ell} - a^{G^c_\ell(A)}$, with $\lambda \in \mathbb{C}$, we get $\det b(\lambda) = \det \tilde{b}(\lambda)$. Moreover, since $G^c_\ell(A)$ and the restriction of $G^c(\hat{A})$ to C^ℓ have the same strongly connected components (see Proposition 4.4), then by Lemma 5.5 again, $\det \tilde{b}(\lambda) = \det(\lambda E_{C_\ell}^{C^\ell} - r^{\ell+1})\lambda^{\# N^{\ell+1}} = \det(-r^\ell)\det(\lambda I - t^\ell)\lambda^{\# N^{\ell+1}}$, which yields:

$$\det b(\lambda) = \det(-r^\ell)\det(\lambda I - t^\ell)\lambda^{\# N^{\ell+1}} \ . \tag{53}$$

Hence, $\det b(\mu) = 0$ since μ is an eigenvalue of t^ℓ, and $\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell, V)}$ has rank $< n$. Since μ is a simple eigenvalue of t^ℓ and $\mu \neq 0$, μ is a simple root of the equation $\det b(\lambda) = 0$. Hence, the partial derivative $\partial_\lambda \det b(\lambda)$, evaluated at $\lambda = \mu$, is non-zero, which implies that there is a subset L of $\{1, \ldots, n\}$, of cardinality $n-1$,

such that $\det(b(\mu)_{L,L}) \neq 0$, which shows that $\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell,V)}$ has rank $n-1$. Thus, (49) has only one non-zero solution, up to a scalar multiple, which implies that all the solutions of (52) are of the form ζw, where $\zeta \in \mathbb{C}$ is such that $|\zeta| = 1$, and w is any solution of (52). Let us pick i such that $w_i \neq 0$. Since all the limit points of $\mathcal{W}_\epsilon$ are of the form ζw, with $|\zeta| = 1$, we get $(\mathcal{W}_\epsilon)_j/(\mathcal{W}_\epsilon)_i \to w_j/w_i$ when $\epsilon \to 0$, and since $\mathcal{V}_\epsilon = \epsilon^{\mathrm{diag}\, V}\mathcal{W}_\epsilon$, we get (50).

6.3. **On the choice of the eigenvector V.** We now show that there is, in some sense, a canonical choice of V in Theorem 6.1. Denote by $C_1^\ell, \dots, C_{\nu_\ell}^\ell$ the critical classes of $\hat{A}_\ell$, and by $C_\ell^1, \dots, C_\ell^{\nu_\ell}$ their restrictions to C_ℓ. By Proposition 4.4, $C_1^\ell, \dots, C_{\nu_\ell}^\ell$ are the strongly connected components of $G^c(\hat{A}) \cap C^\ell \times C^\ell$ and they cover C^ℓ. Moreover, one can deduce from Proposition 4.1, that for $\nu = 1, \dots, \nu_\ell$, C_ℓ^ν is either the empty set or a critical class of the matrix A_ℓ, and that $C_\ell^1 \cup \cdots \cup C_\ell^{\nu_\ell} = C_\ell$. Then, when r^ℓ is invertible, the characteristic polynomial of t^ℓ can be factored as

$$\det(\lambda I - t^\ell) = Q_\ell^1(\lambda) \cdots Q_\ell^{\nu_\ell}(\lambda) \tag{54}$$

where $Q_\ell^\nu(\lambda) = \det(\lambda I - t^\ell_{C_\ell^\nu, C_\ell^\nu})$ if $C_\ell^\nu \neq \varnothing$ and $Q_\ell^\nu(\lambda) = 1$ otherwise. Indeed, taking $G = G^c(\hat{A})$ in (43), using the fact that $C_1^\ell, \dots, C_{\nu_\ell}^\ell$ are the strongly connected components of $G^c(\hat{A}) \cap C^\ell \times C^\ell$, and using the block triangular structure of $\lambda E_{N^\ell} - a^{G^c(\hat{A})}$, we get

$$\begin{aligned} \det(-r^\ell)\det(\lambda I - t^\ell) &= \det(\lambda E_{C_\ell}^{C^\ell} - a^{G^c(\hat{A})}_{C^\ell C^\ell}) \\ &= \prod_{\nu=1}^{\nu_\ell} \det(\lambda E_{C_\ell^\nu}^{C_\nu^\ell} - a^{G^c(\hat{A})}_{C_\nu^\ell C_\nu^\ell}) \\ &= \det(-r^\ell) \prod_{\nu=1,\dots,\nu_\ell,\ C_\ell^\nu \neq \varnothing} \det(\lambda I - t^\ell_{C_\ell^\nu C_\ell^\nu}) \ . \end{aligned} \tag{55}$$

Since r^ℓ is invertible, this shows (54). Thus, if $\mu \neq 0$ is a simple root of $\det(\lambda I - t^\ell)$, there is a unique $\nu \in \{1, \dots, \nu_\ell\}$ such that μ is a root of the polynomial $Q_\ell^\nu(\lambda)$. Denote by $\nu(\mu)$ this index. Let V be an eigenvector of $\hat{A}_\ell$, for instance $V = (\hat{A}_\ell)^*_{\cdot,j}$ with $j \in C^\ell$. By the same arguments as in the proof of (53), one can show that (55) remains valid if we replace $G^c(\hat{A}_\ell)$ by $\mathrm{Sat}(\hat{A}_\ell, V)$. Hence, for any $\nu \neq \nu(\mu)$, $(\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell,V)})_{C_\nu^\ell, C_\nu^\ell}$ is invertible. Moreover, since $\mu \neq 0$, $(\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell,V)})_{N^{\ell+1}, N^{\ell+1}}$ is invertible. One can then deduce, using the block triangular structure of $\mu E_{N^\ell} - a^{\mathrm{Sat}(\hat{A}_\ell,V)}$, that if there is no path from i to $C^\ell_{\nu(\mu)}$ in $\mathrm{Sat}(\hat{A}_\ell, V)$, then $w_i = 0$. In particular, using Proposition 2.6, one deduce that if $V = (\hat{A}_\ell)^*_{\cdot,j}$ with $j \in C^\ell \setminus C^\ell_{\nu(\mu)}$, then there exists a final class C_ν^ℓ of $\mathrm{Sat}(\hat{A}_\ell, V)$ different from $C^\ell_{\nu(\mu)}$, hence $w_i = 0$ for all $i \in C_\nu^\ell$. This observation explains Example 6.3, and it also suggests that the choice $V = (\hat{A}_\ell)^*_{\cdot,j}$ with $j \in C^\ell_{\nu(\mu)}$ is canonical (note that different choices of $j \in C^\ell_{\nu(\mu)}$ yield proportional vectors V). However, in the case of eigenvectors, there does not seem to be a simple analogue of Proposition 5.2 (characterising the cases where generically w has non-zero entries).

7. The theorem of Višik, Ljusternik, and Lidskiĭ revisited

7.1. **Statement of the theorem.** We now show that the theorem of Višik and Ljusternik [VL60] and Lidskiĭ [Lid65] can be obtained as a corollary of Theorem 5.1, and that Theorem 5.1 allows to solve cases to which the classical result does not apply. The presentation of this subsection is inspired by [MBO97], that the reader may consult for a general discussion of the theory of Višik, Ljusternik, and Lidskiĭ.

Lidskiĭ [Lid65] considers a matrix of the form $\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b$, where $b \in \mathbb{C}^{n\times n}$ and $\mathcal{A}_0 \in \mathbb{C}^{n\times n}$ is a nilpotent matrix. We shall need specific notations for Jordan matrices. Let $N[q]$ denote the $q \times q$ nilpotent matrix such that $(N[q])_{i,j} = 1$ if $j = i+1$, and $(N[q])_{i,j} = 0$ otherwise. For $m \geq 1$, we define $N\left[\begin{smallmatrix} m \\ q \end{smallmatrix}\right] = N(q)\dot{+}\cdots\dot{+}N(q)$ (m-times), where $\dot{+}$ denotes the block diagonal sum, and, given a decreasing sequence $q_1 > q_2 > \ldots > q_k \geq 1$, and $m_1, \ldots, m_k \geq 1$, we define $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right] = N\left[\begin{smallmatrix} m_1 \\ q_1 \end{smallmatrix}\right]\dot{+}\cdots\dot{+}N\left[\begin{smallmatrix} m_k \\ q_k \end{smallmatrix}\right]$. For instance, when $q_1 = 3, m_1 = 1, q_2 = 2, m_2 = 2, q_3 = 1, m_3 = 1$, we have

$$N\left[\begin{smallmatrix} 1,2,1 \\ 3,2,1 \end{smallmatrix}\right] = \left[\begin{array}{ccc|cc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot}
\end{array}\right] , \tag{56}$$

where, again, "$\cdot$" represents 0 (why some zero entries are written inside boxes will be explained below). We consider the case where $\mathcal{A}_0$ is equal to $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$. If $1 \leq \ell \leq k$, we finally define the $(m_1 + \cdots + m_\ell) \times (m_1 + \cdots + m_\ell)$ submatrix Φ_ℓ of b, obtained by considering only the bottom rows and first columns of the Jordan cells of sizes $q_i \times q_j$, $i, j = 1, \ldots, \ell$. For instance, in the case of (56),

$$\Phi_1 = \begin{bmatrix} b_{31} \end{bmatrix} , \qquad \Phi_2 = \begin{bmatrix} b_{31} & b_{34} & b_{36} \\ b_{51} & b_{54} & b_{56} \\ b_{71} & b_{74} & b_{76} \end{bmatrix} , \qquad \Phi_3 = \begin{bmatrix} b_{31} & b_{34} & b_{36} & b_{38} \\ b_{51} & b_{54} & b_{56} & b_{58} \\ b_{71} & b_{74} & b_{76} & b_{78} \\ b_{81} & b_{84} & b_{86} & b_{78} \end{bmatrix} .$$

The corresponding positions in the matrix $\mathcal{A}_0$ were depicted by boxes in (56). By convention, Φ_0 is the empty matrix, and is invertible.

Corollary 7.1 ([Lid65, Th. 1])**.** *Assume that both $\Phi_{\ell-1}$ and Φ_ℓ are invertible, for some $1 \leq \ell \leq k$, and let $\lambda_1, \ldots, \lambda_{m_\ell}$ denote the eigenvalues of* $\mathrm{Schur}(\Phi_{\ell-1}, \Phi_\ell)$. *Then, $\mathcal{A}_\epsilon$ has $m_\ell q_\ell$ eigenvalues with asymptotics*

$$\mathcal{L}_\epsilon \sim \xi \epsilon^{1/q_\ell}, \text{ where } \xi^{q_\ell} = \lambda_i \text{ and } i = 1, \ldots, m_\ell$$

(for each λ_i, all the q_ℓ-th roots ξ of λ_i are taken).

Of course, Corollary 7.1 can be stated in an equivalent "coordinate free" way, by using left and right eigenvectors associated to the different Jordan blocks, see [Lid65]. In fact, Moro, Burke, and Overton observed that we need not require Φ_ℓ to be invertible in Corollary 7.1: when Φ_ℓ is singular, [MBO97, Th. 2.1] shows that to each eigenvalue $\lambda_i \in \mathbb{C}$ of $\mathrm{Schur}(\Phi_{\ell-1}, \Phi_\ell)$ corresponds q_ℓ eigenvalues of $\mathcal{A}_\epsilon$ with asymptotics $\mathcal{L}_\epsilon = \xi\epsilon^{1/q_\ell} + o(\epsilon^{1/q_\ell})$ where $\xi^{q_\ell} = \lambda_i$.

7.2. **Derivation of Corollary 7.1.** Let us denote by $A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ the matrix of exponents associated to $\mathcal{A}_\epsilon = N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right] + \epsilon b$: $A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ is obtained from $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ by exchanging zeros and ones. For instance, $A\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ corresponds to $\mathcal{A}_\epsilon = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] + \epsilon b$. The following lemma is straightforward.

Lemma 7.2. *Let $q_1 > \cdots > q_k \geq 1$, $m_1, \ldots, m_k \geq 1$. The matrix $A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ has min-plus eigenvalue $1/q_1$, and set of critical nodes $C_1 = \{1, \ldots, m_1 q_1\}$. Moreover,*

$$\mathrm{Schur}(C_1, 1/q_1, A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]) = A\left[\begin{smallmatrix} m_2\ldots,m_k \\ q_2,\ldots,q_k \end{smallmatrix}\right] \ . \tag{57}$$

It follows from Lemma 7.2 and in particular, from the recursive property (57), that the sequence of critical values of $A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ is $(\alpha_1, \ldots, \alpha_k) = (1/q_1, \ldots, 1/q_k)$, and that the associated critical classes are $C_1 = \{1, \ldots, m_1 q_1\}$, $\ldots$, $C_k = \{\sum_{\ell=1}^{k-1} m_\ell q_\ell + 1, \ldots, \sum_{\ell=1}^{k} m_\ell q_\ell\}$. Recall that the diagonal matrix D is defined from the α_ℓ and C_ℓ.

An eigenvector $V\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ of $D^{-1}A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ can be built as follows. For all $q \geq 1$, we set $V\left[\begin{smallmatrix} 1 \\ q \end{smallmatrix}\right] = [0, 1/q, \ldots, (q-1)/q]^T$, then, for $m \geq 1$, we define $V\left[\begin{smallmatrix} m \\ q \end{smallmatrix}\right] = V\left[\begin{smallmatrix} 1 \\ q \end{smallmatrix}\right] \dot{+} \cdots \dot{+} V\left[\begin{smallmatrix} 1 \\ q \end{smallmatrix}\right]$ (m-times), where $\dot{+}$ denotes the concatenation of vectors, and, finally, we set $V\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right] = V\left[\begin{smallmatrix} m_1 \\ q_1 \end{smallmatrix}\right] \dot{+} \cdots \dot{+} V\left[\begin{smallmatrix} m_k \\ q_k \end{smallmatrix}\right]$. It is easy to see that $V = V\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ is an eigenvector of $\hat{A} = D^{-1}A\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$, and that the corresponding saturation graph is the union of the graph of $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$ and of the arcs (i, j), where i is the index of a bottom row of a Jordan block of $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$, and j is the index of the left column of a Jordan block of $N\left[\begin{smallmatrix} m_1,\ldots,m_k \\ q_1,\ldots,q_k \end{smallmatrix}\right]$. Since $\mathrm{Sat}(\hat{A}, V)$ is strongly connected, it is also equal to $G^c(\hat{A})$. For instance, for $\mathcal{A}_\epsilon = N\left[\begin{smallmatrix} 1,2,1 \\ 3,2,1 \end{smallmatrix}\right] + \epsilon b$, and $G = G^c(\hat{A}) = \mathrm{Sat}(\hat{A}, V)$, we get

$$a^G = \left[\begin{array}{ccc|cc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{31} & \cdot & \cdot & b_{34} & \cdot & b_{36} & \cdot & b_{38} \\ \hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
b_{51} & \cdot & \cdot & b_{54} & \cdot & b_{56} & \cdot & b_{58} \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
b_{71} & \cdot & \cdot & b_{74} & \cdot & b_{76} & \cdot & b_{78} \\ \hline
b_{81} & \cdot & \cdot & b_{84} & \cdot & b_{86} & \cdot & b_{88}
\end{array}\right] . \tag{58}$$

The statement of Corollary 7.1 becomes a special case of the statement of Theorem 5.1, provided the following identity is proved:

$$\det(\lambda I - t^\ell) = \det(\lambda^{q_\ell} - \mathrm{Schur}(\Phi_{\ell-1}, \Phi_\ell)), \quad \ell = 1, \ldots, k \ .$$

This can be seen immediately by noting that t^ℓ is a matrix of cyclicity q_ℓ, which can be put, by applying a transformation $t^\ell \mapsto P_\ell t^\ell P_\ell^{-1}$, for some permutation matrix P_ℓ, in block circular form

$$P_\ell t^\ell P_\ell^{-1} = \begin{bmatrix} \cdot & I_{m_\ell(q_\ell-1)} \\ \mathrm{Schur}(\Phi_{\ell-1}, \Phi_\ell) & \cdot \end{bmatrix} , \tag{59}$$

where I_q is the identity matrix of order q, and where the "$\cdot$" represent blocks with 0 values.

Indeed, by (45) and (43), we get:

$$t^\ell = \mathrm{Schur}(C^{\ell-1}, a^G_{C^\ell C^\ell}) \tag{60}$$

and for each $\ell = 1, \dots, k$, there exists a matrix Q_ℓ corresponding to a permutation of C^ℓ preserving C_ℓ, such that in block form we get:

$$Q_\ell a^G_{C^\ell,C^\ell} Q_\ell^{-1} = \left[\begin{array}{cc|cc} \cdot & I_{m_1(q_1-1)+\cdots+m_{\ell-1}(q_{\ell-1}-1)} & \cdot & \cdot \\ \Phi_\ell^{11} & \cdot & \Phi_\ell^{12} & \cdot \\ \hline \cdot & \cdot & \cdot & I_{m_\ell(q_\ell-1)} \\ \Phi_\ell^{21} & \cdot & \Phi_\ell^{22} & \cdot \end{array}\right] ,$$

where $\Phi_\ell = \left[\begin{smallmatrix} \Phi_\ell^{11} & \Phi_\ell^{12} \\ \Phi_\ell^{21} & \Phi_\ell^{22} \end{smallmatrix}\right]$ and $\Phi_\ell^{11} = \Phi_{\ell-1}$ (for each ℓ, the indices of Φ_ℓ^{22} correspond to the nodes of $C_\ell = \{\sum_{i=1}^{\ell-1} m_i q_i + 1, \dots, \sum_{i=1}^{\ell} m_i q_i\}$ of the form $\sum_{i=1}^{\ell-1} m_i q_i + m q_\ell$ with $m = 1, \dots, m_\ell$). Hence, taking for P_ℓ the restriction of Q_ℓ to C_ℓ, and using the fact that $\left[\begin{smallmatrix} \cdot & \Psi \\ \Phi & \cdot \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \cdot & \Phi^{-1} \\ \Psi^{-1} & \cdot \end{smallmatrix}\right]$ for all invertible matrices Ψ and Φ, we get (59).

For instance, in the special case of (58), and $\ell = 2$, we get

$$Q_2 a^G_{C^2,C^2} Q_2^{-1} = \left[\begin{array}{ccc|cccc} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ b_{31} & \cdot & \cdot & b_{34} & b_{36} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ b_{51} & \cdot & \cdot & b_{54} & b_{56} & \cdot & \cdot \\ b_{71} & \cdot & \cdot & b_{74} & b_{76} & \cdot & \cdot \end{array}\right] ,$$

and,

$$\begin{aligned} P_2 t^2 P_2^{-1} &= \left[\begin{array}{c|cc} \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix} & 1 & \cdot \\ & \cdot & 1 \\ \hline \begin{bmatrix} b_{54} & b_{56} \\ b_{74} & b_{76} \end{bmatrix} - \begin{bmatrix} b_{51} \\ b_{71} \end{bmatrix} b_{31}^{-1} \begin{bmatrix} b_{34} & b_{36} \end{bmatrix} & \cdot & \cdot \\ & \cdot & \cdot \end{array}\right] \\ &= \begin{bmatrix} 0 & I_2 \\ \mathrm{Schur}(\Phi_1, \Phi_2) & 0 \end{bmatrix} . \end{aligned}$$

This concludes the proof of Corollary 7.1.

7.3. **Singular examples.** We now show how Theorem 5.1 allows to solve singular cases in Lidskiĭ's theorem (Corollary 7.1), and we also illustrate the limitations of Theorem 5.1.

Example 7.3. Consider the following classical degenerate example, taken from [Wil65, Section 2.22] and [MBO97, Eqn 1.1]:

$$\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b, \text{ where } \mathcal{A}_0 = N\left[\begin{smallmatrix} 1,1 \\ 3,2 \end{smallmatrix}\right] = \left[\begin{array}{ccc|cc} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \odot & \cdot & \cdot & \boxdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \\ \boxdot & \cdot & \cdot & \boxdot & \cdot \end{array}\right], \text{ and } b \in \mathbb{C}^{n\times n} \quad . \tag{61}$$

Recall that all the dots (whether they are surrounded by boxes or circles, or not) represent 0. If the entry b_{31} corresponding to the circled position in (61), is zero, Φ_1 is singular, and we cannot apply Lidskiĭ's theorem (Corollary 7.1). However,

Theorem 5.1 can be applied. We can write $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$, with

$$a = \left[\begin{array}{ccc|cc} b_{11} & 1 & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & 1 & b_{24} & b_{25} \\ 0 & b_{32} & b_{33} & b_{34} & b_{35} \\ \hline b_{41} & b_{42} & b_{43} & b_{44} & 1 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{array}\right] \text{ , and } A = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \infty & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right] .$$

We have $\alpha_1 = \rho_{\min}(A) = 2/5$, $C_1 = \{1,2,3,4,5\}$, and since the critical graph of A, which is composed only of the circuit $(1 \to 2 \to 3 \to 4 \to 5 \to 1)$ covers all the nodes, we have $G^c(\hat{A}) = G^c(A)$. Thus, for $G = G^c(\hat{A})$,

$$a^G = \left[\begin{array}{ccc|cc} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & b_{45} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \\ b_{51} & \cdot & \cdot & \cdot & \cdot \end{array}\right] .$$

Theorem 5.1 shows that, if $b_{45}b_{51} \neq 0$, $\mathcal{A}_\epsilon$ has five roots with asymptotics

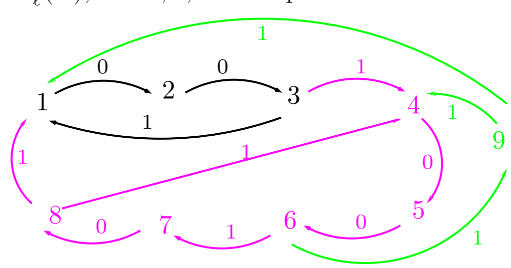
$$\mathcal{L}_\epsilon \sim \xi\epsilon^{2/5} \text{ , where } \xi^5 = b_{45}b_{51} .$$

The asymptotics of the eigenvectors can also be obtained from Theorem 6.1 (the computations are similar to the case of Example 6.2).

Example 7.4. Let us discuss the following singular version of the illustrating example of [MBO97]. Let $\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b$, where $\mathcal{A}_0 = N\left[\begin{smallmatrix}2,1,1\\3,2,1\end{smallmatrix}\right]$, so that, setting $G = G^c(\hat{A})$,

$$a^G = \left[\begin{array}{ccc|ccc|cc|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{31} & \cdot & \cdot & b_{34} & \cdot & \cdot & b_{37} & \cdot & b_{39} \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ b_{61} & \cdot & \cdot & b_{64} & \cdot & \cdot & b_{67} & \cdot & b_{69} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ b_{81} & \cdot & \cdot & b_{84} & \cdot & \cdot & b_{87} & \cdot & b_{89} \\ \hline b_{91} & \cdot & \cdot & b_{94} & \cdot & \cdot & b_{97} & \cdot & b_{99} \end{array}\right] .$$

Consider the singular case where $b_{61} = b_{64} = 0$. We may keep A as in Section 7.2, but this gives little information since t^1 is not invertible. However, $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$ still holds if we change the following values of A: $A_{61} = A_{64} = \infty$. Then, we find $\alpha_1 = 1/3$, $C_1 = \{1,2,3\}$, $\alpha_2 = 2/5$, $C_2 = \{4,5,6,7,8\}$, $\alpha_3 = 4/5$, $C_3 = \{9\}$, and the critical graphs $G^c_\ell(A)$, $\ell = 1,2,3$ are represented as follows:



with the same colouring convention as in Example 4.5.

The matrix t^1 is invertible if, and only if, $b_{31} \neq 0$. In this case, $\mathcal{A}_\epsilon$ has three eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{1/3}$, corresponding to the different cubic roots λ of b_{31}. We have

$$s^2 = \left[\begin{array}{ccc|cc|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b_{67} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ b'_{84} & \cdot & \cdot & \cdot & \cdot & b'_{89} \\ \hline b'_{94} & \cdot & \cdot & \cdot & \cdot & b'_{99} \end{array}\right]$$

where for instance $b'_{84} = b_{84} - b_{81}b_{31}^{-1}b_{34}$. Thus, t^2 is invertible, if, and only if, $b'_{84}b_{67} \neq 0$. When this is the case, $\mathcal{A}_\epsilon$ has five eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{2/5}$, corresponding to the different quintic roots λ of $b'_{84}b_{67}$.

The last critical graph, $G_3^c(A)$, that we just represented above, does not have a disjoint circuit cover. To see this, observe that there is no arc from the set $\{3, 8, 9\}$ to the set $\{2, 3, 5, 6, 7, 8, 9\}$, remark that the sum of the numbers of elements of these two sets, which is $3 + 7 = 10$, exceeds the dimension of the matrix, which is 9, and apply the Frobenius-König theorem (see for instance [BR97, Th. 2.14]). Then, we know from Theorems 4.6 and 4.7 that the greatest root, γ_9, of the min-plus characteristic polynomial of A, P_A, is strictly greater than the greatest critical value, $\beta_9 = \alpha_3 = 4/5$, and by Theorem 3.8, the exponent Λ_9 of the remaining eigenvalue of $\mathcal{A}_\epsilon$ must be strictly greater than $4/5$. Thus, in this case, Theorem 5.1 does not predict the exponent of the eigenvalue of $\mathcal{A}_\epsilon$ of minimal modulus. However, this exponent can be obtained as follows. We already know that $\alpha_1 = 1/3$ and $\alpha_2 = 2/5$ are roots of respective multiplicity 3 and 5 of P_A, so the associated characteristic polynomial function is of the form $\widehat{P}_A(y) = (y \oplus \alpha_1)^3(y \oplus \alpha_2)^5(y \oplus \gamma_9)$. One can check that $\operatorname{per} A = 4$, and since $\hat{P}_A(\mathbb{0}) = \operatorname{per} A$, we deduce that $\alpha_1^3\alpha_2^5\gamma_9 = 3\gamma_9 = 4$, therefore, $\gamma_9 = 1$. Then, one can derive from Theorem 3.8 that $\Lambda_9 = 1$, for generic values of b. The problem of finding the leading coefficient of the corresponding eigenvalue of $\mathcal{A}_\epsilon$ is solved by the result of [ABG04].

Example 7.5. Corollary 3.3 of [MBO97] identifies a special situation where the leading exponent of a group of eigenvalues can be found although the corresponding matrix $\Phi_{\ell-1}$ appearing in Lidskiĭ's theorem (see Corollary 7.1 above) is not invertible. We next give an example which cannot be solved using the method of [MBO97] but which is solved by Theorem 5.1. Let

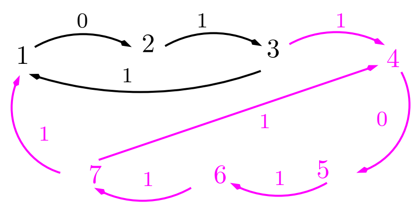
$$\mathcal{A}_\epsilon = \left[\begin{array}{cc|c|cc|c|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \epsilon b_{23} & \cdot & \cdot & \cdot & \cdot \\ \hline \epsilon b_{31} & \cdot & \cdot & \epsilon b_{34} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon b_{56} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon b_{67} \\ \hline \epsilon b_{71} & \cdot & \cdot & \epsilon b_{74} & \cdot & \cdot & \cdot \end{array}\right],$$

where the b_{ij} are complex numbers, and, as above, the dots represent zero entries. The matrix $\mathcal{A}_\epsilon$ is of the form $\mathcal{A}_0 + \epsilon b$, where the matrix $\mathcal{A}_0$ is a nilpotent matrix

conjugate to $N\left[\begin{smallmatrix}2,3\\2,1\end{smallmatrix}\right]$. The corresponding matrix A is

$$A = \left[\begin{array}{cc|c|cc|c|c} \infty & 0 & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 1 & \infty & \infty & \infty & \infty \\ \hline 1 & \infty & \infty & 1 & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 1 & \infty \\ \hline \infty & \infty & \infty & \infty & \infty & \infty & 1 \\ \hline 1 & \infty & \infty & 1 & \infty & \infty & \infty \end{array}\right] .$$

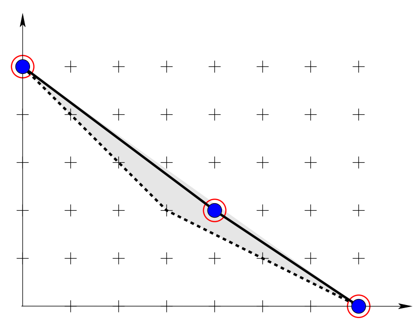
We have $\alpha_1 = 2/3$, $C_1 = \{1, 2, 3\}$, $\alpha_2 = 3/4$, $C_2 = \{4, 5, 6, 7\}$. The critical graphs $G^c_1(A)$ and $G^c_2(A)$ are the following:



with the same colouring convention as in the previous examples. Theorem 5.1 shows that, for generic values of the coefficients b_{ij}, the matrix $\mathcal{A}_\epsilon$ has three eigenvalues $\mathcal{L}_\epsilon \sim \lambda\epsilon^{2/3}$, where λ is a cubic root of $b_{23}b_{31}$, and four eigenvalues of the form $\lambda\epsilon^{3/4}$, where λ is a quartic root of

$$b_{56}b_{67}(b_{74} - b_{71}b_{31}^{-1}b_{34}) .$$

The following picture represents the actual Newton polygon of the characteristic polynomial $\det(\mathsf{Y}I - \mathcal{A}_\epsilon)$, for generic values of the b_{ij} (this is exactly the graph of $\overline{P}$, where $P = \mathrm{per}(\mathsf{Y}I \oplus A)$.) A monomial $\mathsf{Y}^i\epsilon^j$ is represented by the point of coordinates (i, j). Integer points are represented by small crosses. The actual Newton polygon (black broken line) consists of two segments of respective slopes $-3/4$ and $-2/3$, joining the three circles. The approximation of the Newton polygon provided by Lidskiĭ's theorem is given by the dashed broken line.



The method of [MBO97] relies on the observation that Lidskiĭ's theorem provides an approximation of the Newton polygon, which is exact when the matrices Φ_ℓ are invertible. Corollary 3.3 of [MBO97] requires the absence of integers points strictly between Lidskiĭ's approximation and its chord, i.e. in the present case, in the interior of the gray region. Since this interior contains the integer point $(4, 2)$, the leading exponents $2/3$ and $3/4$ cannot be obtained from [MBO97].

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